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Classification of Complex Semisimple Lie Algebras

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Abstract

In this report we discuss the classification of complex semisimple lie algebras. The classification relies on the bijective correspondence between complex semisimple Lie algebras and root systems: in particular the correspondence between complex simple Lie algebras and irreducible root systems and the similar roles these two structures play in the construction of complex semisimple Lie algebras and reducible root systems respectively. The final result is very neat, classifying all complex simple Lie algebras as one of the following classes, $\mathbf{A_n}$ $(n \geq 1)$, $\mathbf{B_n}$ $(n \geq 2)$, $\mathbf{C_n}$ $(n \geq 3)$, $\mathbf{D_n}$ $(n \geq 4)$, $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$, $\mathbf{F_4}$ or $\mathbf{G_2}$, and complex semisimple Lie algebras as direct sums of such Lie algebras.

Career Development and Acknowledgements.

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Research Summary.

A Lie algebra is a vector space $\mathfrak{g}$ endowed with additional structure through a bilinear operator called a commutator $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. It is a fair enough question to ask whether the resulting structure can be enumerated and fully classified. Due to the difficulty in this question we restrict our focus to semisimple Lie algebras over $\mathbb{C}$. The full classification of complex semisimple Lie algebras, as we know it today, was given by a young Evgeniĭ B. Dynkin in 1947 (Dynkin, 1947) and later adapted by Jean-Pierre Serre to give us a complete correspondence between Lie algebras and their root systems up to isomorphism, as outlined in (Serre, 2012). Books such as (Humphreys, 2012) and (Knapp, 2013) give slightly different methods of decomposition and classification of abstract root systems, however the fundamentals remain the same. That is, given a complex semisimple Lie algebra $\mathfrak{g}$ we can decompose $\mathfrak{g}$ into the direct sum of a collection of 'simultaneous eigenspaces' relative to the adjoint representation of a Cartan subalgebra $\mathfrak{h}$: $\operatorname{ad}\mathfrak{h}$ being maximal abelian and simultaneously diagonalisable, ensuring we can decompose $\mathfrak{g}$ in its entirety. To each eigenspace corresponds an element of $\mathfrak{h}^*$ we call a root. The collection of these roots Φ form a complex root system in $\mathfrak{h}^*$. The process is often referred to as the root space decomposition of $\mathfrak{g}$ with respect to Cartan subalgebra $\mathfrak{h}$. It can then be shown that any two Cartan subalgebras $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are isomorphic – with $\mathfrak{h}_1 = [g, \mathfrak{h}_2]$ for some $g \in \mathfrak{g}$ – and that as consequence the resultant root system Φ is independent of the choice of $\mathfrak{h}$. Thus each Lie algebra $\mathfrak{g}$ has a unique complex root system $\Phi_{\mathfrak{g}}$. J-P.Serre showed that two Lie algebras with isomorphic root systems must in turn be isomorphic and that to each abstract root system Φ there exists a complex semisimple Lie algebra whose root system is $\Phi_{\mathfrak{g}}$ with $\Phi = \Phi_{\mathfrak{g}}$ (Isomorphism and Existence theorems). This then gives a bijective correspondence between the two structures. However each complex abstract root system is simply the complexification of a real abstract root system i.e. Φ is a real root system on the $\mathbb{R}$ span of Φ in $\mathfrak{h}^*$. So in classifying all real root systems up to isomorphism we have gained a classification of complex root systems and in turn, thanks to the isomorphism an existence theorems, classified all complex semisimple Lie algebras. We will not give an outline of the classification of real root systems here but will instead give the results of the classification, those being that each real root system is one of four infinite families or five exceptional root systems: $\mathbf{A_n}$ $(n \geq 1)$, $\mathbf{B_n}$ $(n \geq 2)$, $\mathbf{C_n}$ $(n \geq 3)$, $\mathbf{D_n}$ $(n \geq 4)$, $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$, $\mathbf{F_4}$ or $\mathbf{G_2}$. The indexing is to remove repeats, such that each root system listed above is distinct up to isomorphism. The report does not aim to prove any results, instead giving a brief overview of the classification. Refer to Humpheys, Knapp, and Serre as listed in the References for proofs on the subject.

Results and Methods.

1 Introduction to Lie Algebras.

1.1 Lie Algebras.

In attempting to classify all semisimple Lie algebras over $\mathbb{C}$ we naturally become interested in the building blocks of such Lie algebras. In this section we discuss what a Lie algebra is; isomorphisms and representations of Lie algebras; and semisimple Lie algebras.

Definition. A **Lie algebra** is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$, endowed with a bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (often called the **bracket** or **commutator**) with the following properties. For all $x, y, z \in \mathfrak{g}$

[L1] $[x, y] = -[y, x]$ (*Skew-symmetry*)

[L2] $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ (*Jacobi identity*)

If $\mathfrak{a}$ is a subspace of a Lie algbera $\mathfrak{g}$, that is closed under the bracket $[\ ,\ ]$ (i.e. $[a, b] \in \mathfrak{a}$ for all $a, b \in \mathfrak{a}$), then we call $\mathfrak{a}$ a **Lie subalgebra** of $\mathfrak{g}$ (or typically a subalgebra of $\mathfrak{g}$).

Examples.

1. An example of a Lie algebra the reader is already well acquainted with is that of $\mathbb{R}^3$ endowed with the cross-product $u \times v$. In particular, we let $\mathfrak{g} = \mathbb{R}^3$ and define the bracket to be exactly the cross-product with $[u, v] = u \times v$. [L1] is satisfied as the cross-product is well know to be bilinear and skew symmetric, where $u \times v = -v \times u$ for $u, v \in \mathbb{R}^3$. We also note that for all $u, v, w \in \mathbb{R}^3$

$$\begin{aligned} u \times (v \times w) &= (u \times v) \times w + v \times (u \times w) \\ &= -w \times (u \times v) - v \times (w \times u). \end{aligned}$$

   And so [L2] (called the **Jacobi identity**) is satisfied, i.e.

$$u \times (v \times w) + w \times (u \times v) + v \times (w \times u) = 0 \quad \text{for all} \quad u, v, w \in \mathbb{R}^3$$

   So $(\mathbb{R}^3, \times)$ does indeed constitute a Lie algebra.

2. A family of linear Lie algebras of particular interest are the **special linear Lie algebras** denoted by $\mathfrak{sl}(n, \mathbb{C})$ whose elements are n by n traceless matrices with complex entries and bracket $[A, B] = AB - BA$. For example $\mathfrak{sl}(2, \mathbb{C})$ is a three-dimensional complex vector space with basis elements

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

In any Lie algebra $\mathfrak{g}$ for x, y in $\mathfrak{g}$ we say that x, y **commute** if $[x, y] = 0$. We say that $\mathfrak{g}$ is **abelian** if for all $x, y \in \mathfrak{g}$, x and y commute (or equivalently if $[\mathfrak{g}, \mathfrak{g}] = \{[x, y] : x, y \in \mathfrak{g}\} = \{0\}$). Thanks to the skew-symmetry of the commutator we have that for any $x \in \mathfrak{g}$, $[x, x] = -[x, x]$ and thus $2[x, x] = 0$. Assuming the base field $\mathbb{F}$ of $\mathfrak{g}$ is of characteristic other than 2, we must have $[x, x] = 0$ i.e. x commutes with itself (for this reason, in the future, we will always assume $\mathbb{F}$ has characteristic other than 2). If $\mathfrak{g}$ is a one dimensional Lie algebra with basis $\{a\}$, then we see that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathbb{C}[a, a] = \{0\}$, and so any one dimensional Lie algebra is abelian.

1.2 Isomorphisms, Representations and Ideals.

A natural notion of equivalence of Lie algebras arises by considering structure preserving maps between Lie algebras: the structure of interest being that imposed on the Lie algebra as a vector space by the bracket.

Definition. Let $\mathfrak{a}, \mathfrak{b}$ be Lie algebras, and $\pi : \mathfrak{a} \to \mathfrak{b}$ a linear transformation, then π is a **homomorphism** of Lie algebras if $\pi([x, y]) = [\pi(x), \pi(y)]$ for all $x, y \in \mathfrak{a}$. Let $\ker \pi = \{x \in \mathfrak{a} : \pi(x) = \mathbf{0}\}$ and $\operatorname{im} \pi = \{y \in \mathfrak{b} : \exists x \in \mathfrak{a}, \pi(x) = y\}$. If $\ker \pi = \{0\}$ and $\operatorname{im} \pi = \mathfrak{b}$ then we call π an **isomorphism** of Lie algebras and say that $\mathfrak{a}$ and $\mathfrak{b}$ are **isomorphic**.

We can partition the set of all Lie algebras into isomorphic classes, as isomorphism defines an equivalence relation on Lie algebras. Now we discuss representations of Lie algebras.

Definition′. Let $\mathfrak{g}$ be a Lie algebra and V a vector space over field $\mathbb{F}$. Then a **representation** of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$.

Thus, if $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$ on V then one must have

$$\pi([x, y]) = [\pi(x), \pi(y)] = \pi(x)\pi(y) - \pi(y)\pi(x)$$

for all $x, y \in \mathfrak{g}$, as a result of the definition of the bracket on $\mathfrak{gl}(V)$.

Examples.

1. An important example of a representation is the **adjoint representation**. Given a Lie algebra $\mathfrak{g}$ define the map $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}_{\mathbb{F}} \mathfrak{g} : x \mapsto \operatorname{ad} x$ where the operator $\operatorname{ad} x$ is defined to be such that $\operatorname{ad} x(y) = [x, y]$ for all $y, x \in \mathfrak{g}$. We often write $\operatorname{ad} \mathfrak{g} := \operatorname{im} \operatorname{ad}$. All we must show is that ad is a homomorphism. Due to linearity of the bracket, the adjoint representation is linear on $\mathfrak{g}$ i.e.

   $$\operatorname{ad}(ax + by) = [ax + by, -] = a[x, -] + b[y, -] = a \operatorname{ad} x + b \operatorname{ad} y \quad \text{for all} \quad a, b \in \mathbb{F},\ x, y \in \mathfrak{g}.$$

   We also have that ad preserves the bracket i.e. due to the Jacobi identity, for all $x, y, z \in \mathfrak{g}$

   $$\operatorname{ad}[x, y](z) = [[x, y], z] = [x, [y, z]] - [y, [x, z]] = (\operatorname{ad} x \circ \operatorname{ad} y - \operatorname{ad} y \circ \operatorname{ad} x)(z) = [\operatorname{ad} x, \operatorname{ad} y](z).$$

   Thus, it is now clear that ad is a Lie algebra homomorphism. Moreover, it is a representation.

2. For an example we can easily calculate a basis of the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$. It is a simple exercise to check

   $$[h, e] = 2e, \quad [e, f] = h, \quad \text{and} \quad [h, f] = -2f.$$

   Setting $B = \{e, h, f\}$ as a basis of $\mathfrak{sl}(2, \mathbb{C})$, one has

   $$[\operatorname{ad} e]_B = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\operatorname{ad} h]_B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad [\operatorname{ad} f]_B = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Definition′. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ a Lie subalgebra. Then $\mathfrak{a}$ is said to be an **ideal** if, for all $a \in \mathfrak{a}$ one has

$$[g, a] \in \mathfrak{a} \quad \text{and} \quad [a, g] \in \mathfrak{a}, \quad \text{for all } g \in \mathfrak{g}$$

For those unfamiliar, an ideal can be thought of as part of an algebra that contributes to the algebraic structure disjointly i.e it can be removed and leave behind another algebra.

1.3 Simple and Semisimple Lie Algebras.

We now look at the building blocks of semisimple Lie algebras and what it means to be semisimple.

Definition. Let $\mathfrak{g}$ be a Lie algebra, then we say that $\mathfrak{g}$ is **simple** if

i. $\mathfrak{g}$ is non abelian;

ii. $\mathfrak{g}$ has no non-zero ideals apart from $\mathfrak{g}$.

These algebras can be thought of as bricks. Due to the lack of non-trivial ideals, there is no interesting way to decompose simple Lie algebras into smaller algebras.

Definition′. Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is said to be **semisimple** if there exists ideals $\mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_n$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \bigoplus_i \mathfrak{a}_i$$

with each $\mathfrak{a}_i$ simple.

Indeed, the building block analogy holds as we can represent any semisimple Lie algebra as the sum of simple Lie algebras. This gives us the ability to reorient our focus to classifying all simple Lie algebras over $\mathbb{C}$, as we can naturally extend the classification to semisimple Lie algebras.

2 Cartan Subalgebras and Root-space Decomposition.

In this section we consider how one gains a root system from a finite dimensional complex semisimple Lie algebra.

2.1 Weight-space Decomposition.

We introduce the machinery required to decompose a complex semisimple Lie algebra. Let $\mathfrak{h}$ be a finite dimensional Lie algebra over $\mathbb{C}$.

Definition. Let π be a representation of $\mathfrak{h}$ on a complex vector space V. For $\alpha \in \mathfrak{h}^*$ we define

$$V_\alpha := \{v \in V : (\pi(h) - \alpha(h)\mathbf{1})^n v = 0 \text{ for all } h \in \mathfrak{h} \text{ for some } n = n(h, v) \in \mathbb{N}\}$$

If $V_\alpha \neq 0$, we call V_α a **generalised weight-space**, and we call α a **weight**. Note that n above is dependent on both $h \in \mathfrak{h}$ and $v \in V$.

We find it useful to mention the following. Let $\mathfrak{h}$ be a Lie algebra, and suppose $\{\operatorname{ad} h_1 \circ \operatorname{ad} h_2 \circ \cdots \circ \operatorname{ad} h_n : h_i \in \mathfrak{h}\} = \{\mathbf{0}\} \subseteq \operatorname{End}_{\mathbb{F}} \mathfrak{h}$ for some $n \in \mathbb{N}$, then we say that $\mathfrak{h}$ is **nilpotent**. For example, any Lie algebra consisting of strictly upper triangular matrices with bracket $[A, B] = AB - BA$ is nilpotent. Note that any abelian Lie algebra is nilpotent: as if $[\mathfrak{h}, \mathfrak{h}] = 0$, then $\operatorname{ad} h = \mathbf{0}$ for all $h \in \mathfrak{h}$. We then observe the properties of the following weight-space decomposition.

Proposition. Let $\mathfrak{h}$ be a nilpotent Lie algebra over $\mathbb{C}$ and π a representation of $\mathfrak{h}$ on a finite dimensional complex vector space V. Then there are finitely many generalised weights α_i, each V_{α_i} is stable under $\pi(\mathfrak{h})$ and we can write V as the direct sum of its generalised weight-spaces i.e.

$$V = \bigoplus_i V_{\alpha_i}.$$

2.2 Cartan Subalgebras.

Definition. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is said to be a **Cartan subalgebra** of $\mathfrak{g}$ if $\mathfrak{h} = \mathfrak{g}_0$.

Theorem (Existence)**.** Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ has a Cartan subalgebra $\mathfrak{h}$.

Theorem′ (Isomorphism)**.** Let $\mathfrak{g}$ be a Lie algebra with $\mathfrak{h}_1$ and $\mathfrak{h}_2$ as Cartan subalgebras of $\mathfrak{g}$. Then there exists an inner automorphism of $\mathfrak{g}$ (denote it π), with $\mathfrak{h}_1 = \pi(\mathfrak{h}_2)$. Namely, $\mathfrak{h}_1 \cong \mathfrak{h}_2$, with $\mathfrak{h}_1 = [g, \mathfrak{h}_2]$ for some $g \in \mathfrak{g}$.

So a Cartan subalgebra is guaranteed to exist and is isomorphic by conjugation to other Cartan subalgebras of the same underlying Lie algebra. We then note the following theorem.

Theorem″. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ a subalgebra. If $\mathfrak{g}$ is semisimple, then $\mathfrak{h}$ is a Cartan subalgebra if $\mathfrak{h}$ is maximal abelian and $\mathrm{ad}_{\mathfrak{g}}\mathfrak{h}$ is simultaneously diagonalisable.

2.3 Root-space Decomposition.

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then we can represent $\mathfrak{h}$ by its adjoint action on $\mathfrak{g}$, we denote by $\mathrm{ad}_{\mathfrak{g}}\,\mathfrak{h}$. Using the weight-space decomposition mentioned previously, we can decompose $\mathfrak{g}$ into generalised weight-spaces with respect to the representation $\mathrm{ad}_{\mathfrak{g}}$ of $\mathfrak{h}$ on $\mathfrak{g}$.

In consideration of Proposition 2.1, we denote the finite collection of non-zero weights by $\Phi_{\mathfrak{g}}$; we then write $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{g}}} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} : (\mathrm{ad}_{\mathfrak{g}} h - \alpha(h)\mathbf{1})^n g = 0 \text{ for all } h \in \mathfrak{h} \text{ for some } n = n(h, v)\}$$

Note that $\Phi_{\mathfrak{g}} \subseteq \mathfrak{h}^*$.

3 Abstract Root Systems.

Our next goal is to understand the structure of abstact root systems.

3.1 Reflections.

A Euclidean space is simply a real vector space E endowed with an inner product (,). A reflection in some Euclidean space E is any linear transformation that fixes some hyperplane P in E and sends those vectors orthogonal to P to their negative. Given any vector α in E we can define a reflection σ_α, that fixes the hyperplane $P_\alpha = \{x \in E : (\alpha, x) = 0\}$ and sends $\alpha \mapsto -\alpha$. We can easily show that σ_α is orthogonal and preserves the inner product. We may write $E = P_\alpha \oplus \mathbb{R}\alpha$, and for any $x \in E$ we can write $x = x' + x''$ with $x' \in P_\alpha$ and $x'' \in \mathbb{R}\alpha$. Then

$$\begin{aligned}(\sigma_\alpha(x), \sigma_\alpha(y)) &= (\sigma_\alpha(x' + x''), \sigma_\alpha(y' + y'')) \\ &= (-x' + x'', -y' + y'') \\ &= (x', y') + (x'', y'') \\ &= (x' + x'', y' + y'') \\ &= (x, y)\end{aligned}$$

for all $x, y \in E$, and σ_α is indeed orthogonal. Explicitly, $\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ for $\beta, \alpha \in E$. For convenience we write $\langle \beta, \alpha \rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$.

3.2 Abstract Root Systems.

Now that we have some idea of what a reflection is, we can easily define a root system.

Definition. Let Φ be a set of vectors in E. Then we say that Φ is a **root system** in E if the following are satisfied:

- **[R1]** Φ is finite, spans E, and does not contain $\mathbf{0}$;
- **[R2]** If $\alpha \in \Phi$, then Φ is invariant under the action of σ_α;
- **[R3]** If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.
- **[R4]** If $\alpha \in \Phi$, then the only scalar multiples of α in Φ are $\pm\alpha$.

If Φ is a root system, we call $\alpha \in \Phi$ a **root**.

Definition′. Given an abstract root system Φ we denote the collection of all reflections σ_α with $\alpha \in \Phi$ by W. We call W the Weyl group of Φ.

Definition″. Let Φ be a root system in a real inner product space E. If $\Phi = \Phi' \cup \Phi''$ with $(x, y) = 0$ for all $x \in \Phi'$, $y \in \Phi''$ then we say Φ is **reducible**. We say Φ is **irreducible** if it is not reducible.

There is a natural notion of isomorphism between root systems. If Φ is a root system in E, and Φ' a root system in E' then we say that the two root systems are isomorphic if there exists a vector space isomorphism $\pi : E \to E'$ such that $\pi(\Phi) = \Phi'$, and $\langle \pi(\beta), \pi(\alpha) \rangle = \langle \beta, \alpha \rangle$.

The following theorems show an important correspondence between finite dimensional complex semisimple Lie algebras. Not only can we associate a root system to each complex semisimple Lie algebra, we notice that irreducible root systems play a similar role in the construction of reducible root systems as simple Lie algebras play in the construction of semisimple Lie algebras.

Theorem. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. Then the set of non-zero weights $\Phi_\mathfrak{g}$ spans a real vector space $V_0 = \mathrm{span}_\mathbb{R}\Phi_\mathfrak{g}$, and $\Phi_\mathfrak{g}$ forms a root system on V_0.

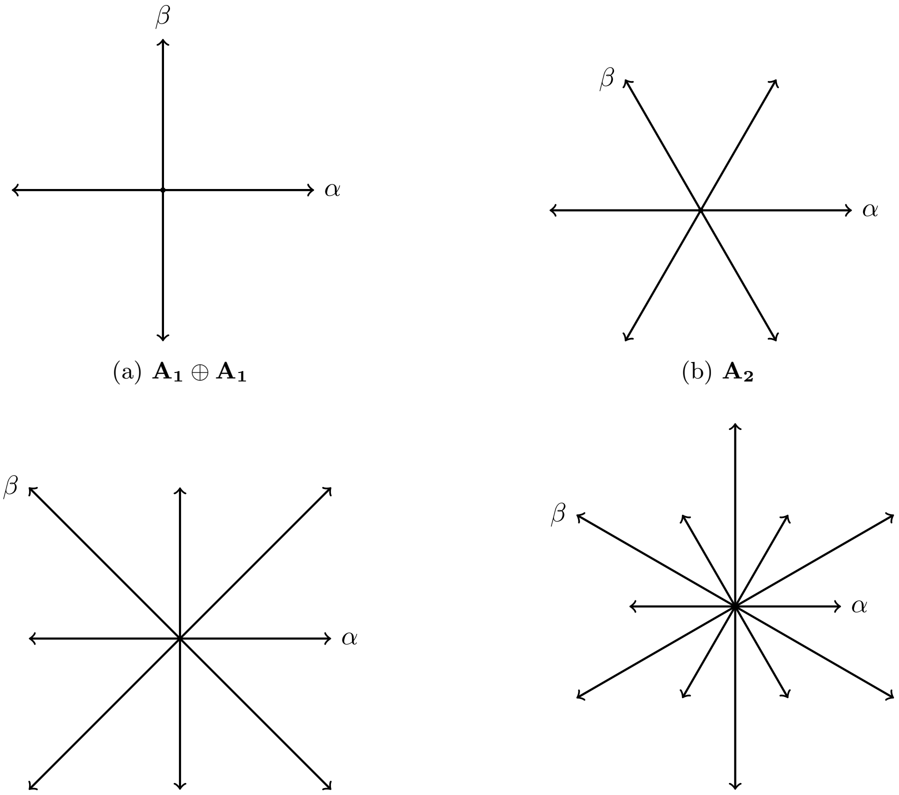
Theorem′. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra and $\Phi_\mathfrak{g}$ its associated root system. Then $\Phi_\mathfrak{g}$ is irreducible if and only if $\mathfrak{g}$ is simple.

Examples.

1. Call $\ell = \dim E$ the **rank** of a root system in E. Thanks to property R4 we only have one reduced root system of rank 1 (up to isomorphism) we label the root system A_1.

$$A_1: \qquad -\alpha \longleftarrow\!\!\!\!\cdot\!\!\!\!\longrightarrow \alpha$$

2. The following are some examples of rank 2 reduced root systems (it turns out there are only four!) Note that $\mathbf{A_1} \oplus \mathbf{A_1}$ is reducible and all other root system are irreducible.



(a) $\mathbf{A_1} \oplus \mathbf{A_1}$ (b) $\mathbf{A_2}$

(c) $\mathbf{B_2}$ (d) $\mathbf{G_2}$

Figure 1: Examples of reduced root systems of rank 2.

3.3 Pairs of Roots.

An important consequence of [R3] is the restriction it places on the relative positioning and length between two roots. Let Φ be a root system on E. Let $\alpha, \beta \in \Phi$, and we assume that $|\alpha|^2 \leq |\beta|^2$ and that α and β are linearly independent. Let θ denote the angle between the two roots with $0 < \theta < \pi$ ($\theta \neq 0$ and $\theta \neq \pi$ due to linear independence). Now, one has

$$\langle \beta, \alpha \rangle = 2\frac{(\beta, \alpha)}{|\alpha|^2} = 2\frac{|\beta|}{|\alpha|}\cos\theta.$$

In particular, we have

$$0 \leq \langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta \leq 4$$

Note that $\langle \beta, \alpha \rangle \leq 0$ if and only if $(\beta, \alpha) \leq 0$, and $\langle \alpha, \beta \rangle \leq 0$ if and only if $(\alpha, \beta) \leq 0$. Since $(\alpha, \beta) = (\beta, \alpha)$ we can conclude that $\langle \beta, \alpha \rangle$ and $\langle \alpha, \beta \rangle$ have the same sign. Moreover, $\langle \beta, \alpha \rangle = 0$ if and only if $\langle \alpha, \beta \rangle = 0$, by a similar consideration. Now since $\theta \neq 0$ and $\theta \neq \pi$,

$$0 \leq \langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta < 4$$

By [R3], $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle \in \mathbb{Z}$, so $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 0, 1, 2,$ or 3.

Suppose $4\cos^2\theta = \langle\beta,\alpha\rangle\langle\alpha,\beta\rangle = 1$ with $\langle\beta,\alpha\rangle = \langle\alpha,\beta\rangle = -1$. Then either $\theta = \pi/3$ or $\theta = 2\pi/3$, however since $\cos(\pi/3) > 0$, if $\theta = \pi/3$ then

$$\langle\beta,\alpha\rangle = 2\frac{|\beta|}{|\alpha|}\cos\theta \geq 0$$

So it cannot be the case that $\theta = \pi/3$ when $\langle\beta,\alpha\rangle = -1$. Hence, it must be the case that $\theta = 2\pi/3$. We also note that

$$\frac{\langle\beta,\alpha\rangle^2}{4\cos^2\theta} = \frac{|\beta|^2}{|\alpha|^2}$$

Thus we see that given our assumptions one has $|\beta|^2/|\alpha|^2 = 1$.

We list all possible values of$\langle\beta,\alpha\rangle$ and $\langle\alpha,\beta\rangle$ (assuming $|\alpha|^2 < |\beta|^2$) alongside the corresponding angular displacements θ and relative lengths $|\beta|^2/|\alpha|^2$ (found by similar considerations as above in the case of $\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle = 1$ and $\langle\beta,\alpha\rangle = \langle\alpha,\beta\rangle = -1$).

$\langle\beta,\alpha\rangle\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	$\langle\alpha,\beta\rangle$	θ	$\lvert\beta\rvert^2/\lvert\alpha\rvert^2$
0	0	0	$\pi/2$	-
1	1	1	$\pi/3$	1
1	-1	-1	$2\pi/3$	1
2	2	1	$\pi/4$	2
2	-2	-1	$3\pi/4$	2
3	3	1	$\pi/6$	3
3	-3	-1	$5\pi/6$	3

Table 1: Relations between pairs of roots.

The following proposition will be useful in later discussion.

Proposition. Let Φ be a reduced root system and $\alpha, \beta \in \Phi$ be linearly independent roots. If $(\alpha,\beta) > 0$, then $\alpha - \beta$ is also a root.

Proof. Note that $(\alpha,\beta) > 0$ if and only if $\langle\alpha,\beta\rangle = 2(\alpha,\beta)/|\beta|^2 > 0$. By observation of table 1, we see that either $\langle\alpha,\beta\rangle = 1$ or $\langle\beta,\alpha\rangle = 1$. In the former case one would have $\sigma_\beta(\alpha) = \alpha - \beta \in \Phi$, and in the latter $\sigma_\alpha(-\beta) = -\beta + \alpha \in \Phi$. In either case $\alpha - \beta$ is a root. □

4 Classification of Abstract Root Systems.

We now state the classification theorem of roots systems via the use of Dynkin diagrams.

4.1 Bases and Simple Roots.

Definition. Let Φ be an abstract root system over E and let $\Delta \subseteq \Phi$. Then Δ is said to be a **base** of Φ is

[B1] Δ is a basis of E;

[B2] and for each $\alpha \in \Phi$ we can write can $\alpha = \sum_i k_i\beta_i$, $\beta_i \in \Delta$ with either all $k_i \geq 0$ or all $k_i \leq 0$.

If Δ is a base, then we say that α is **simple** if $\alpha \in \Delta$.

The first thought that may come to mind is how do we guarantee the existence of a base. But it turns out that a base does exist for any given root system and is unique up to reflection by an element of the Weyl group. We summarise this in the following theorem.

Theorem. Every abstract root system has a base Δ. Moreover, if Δ_1 and Δ_2 are two bases of a root system with Weyl group W, then there exists a $\sigma \in W$ such that $\Delta_1 = \sigma(\Delta_2)$.

We also have the following properties that will be important in the construction of Dynkin diagrams.

Proposition. Let Δ be a base of a root system Φ. If α, β are linearly independent simple roots, then we must have one of the following

1. $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 0$, and $\theta = \pi/2$;
2. $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 1$ and $\theta = 2\pi/3$;
3. $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 2$ and $\theta = 3\pi/4$;
4. $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 3$ and $\theta = 5\pi/6$.

Examples.

1. The roots labled α and β form a base for each root system in Figure 1. By Proposition 4.1, these four root systems are indeed the only root systems of rank 2.

4.2 Dynkin Diagrams.

Definition. Let Φ be a root system with base Δ. A **Coxeter diagram** is a graph, with simple roots for vertices and the number of edges between simple roots α and β given by the value $\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle$. A **Dynkin diagram** is a Coxeter diagram with directed edges, pointing towards the simple root of least magnitude. If two simple roots have the same magnitude the edge remains is un-directed.

Theorem. If Φ is an irreducible root system, then it's associated Dynkin Diagram must be one of the following

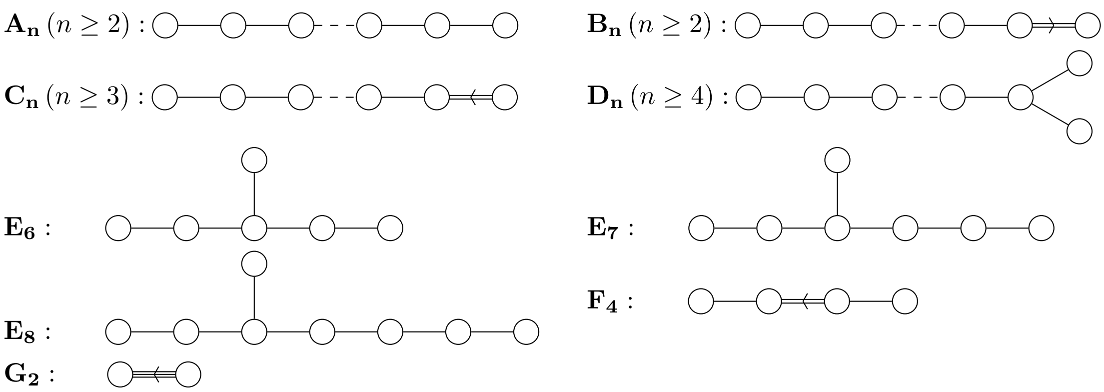


Figure 2: The Classification of root systems.

This restricts the structure of irreducible root systems. Note that the Dynkin diagram of a complex semisimple Lie algebra is a disconnected graph whose components are Dynkin diagrams of irreducible root systems.

5 Isomorphism and Existence Theorems.

We now have a complete classification of abstract root systems. Thus the root system generated by the root-space decomposition of some Lie algebra $\mathfrak{g}$, being an abstract root system, is classified accordingly. However we still wish to establish a bijective correspondence between isomorphic abstract root systems and isomorphic complex semisimple Lie algebras. To do this we have two important theorems given by J-P. Serre. To prove these theorems we require an understanding of how to gain information about the Lie algebra from its associated root system which is beyond the scope of this report.

Theorem (Isomorphism)**.** Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be complex semisimple Lie algebras with root systems Φ_1 and Φ_2. If Φ_1 and Φ_2 are isomorphic as root systems, then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic as Lie algebras.

Simply put, two complex semisimple Lie algebras with isomorphic root systems are isomorphic.

Theorem′ (Existence)**.** Let Φ be a an abstract root system. Then there exists a complex semisimple Lie algebra $\mathfrak{g}$ with root system Φ.

Hence, with these two theorems, we have established a bijective correspondence between finite dimensional complex semisimple Lie algebras and abstract root systems up to isomorphim. The classification of finite dimensional complex semisimple Lie algebras then follows as the structure of abstract root systems is limited to a Dynkin diagram with components listed in Theorem 4.2. We can impose an equivalence relation on all finite dimensional semisimple Lie algebras over $\mathbb{C}$ by considering isomorphism classes. We then associate to each isomorphism class a Dynkin diagram. Note the following about complex simple Lie algebras.

Theorem″. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. If $\mathfrak{g}$ is simple then it must be in one of the following classes: $\mathbf{A_n}\,(n \geq 1)$, $\mathbf{B_n}\,(n \geq 2)$, $\mathbf{C_n}\,(n \geq 3)$, $\mathbf{D_n}\,(n \geq 4)$, $\mathbf{E_6}$, $\mathbf{E_7}$, $\mathbf{E_8}$, $\mathbf{F_4}$ or $\mathbf{G_2}$. Moreover, each class is non-empty.

The classification then naturally extends to semisimple Lie algebras over $\mathbb{C}$: such algebras being in equivalence classes represented by the direct sums of simple Lie algebras over $\mathbb{C}$ i.e. $\mathbf{A_1} \oplus \mathbf{B_2}$ is an equivalence class of semisimple complex Lie alegbras with reducible root systems isomorphic to $\mathbf{A_1} \cup \mathbf{B_2}$.

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