# An Analytic Proof of the Morse Inequalities

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# Abstract

Morse theory is the study of the relationship between the critical points of a class of functions on a closed $n$-manifold $M$ and the topology of the manifold. In this dissertation we state and prove the Morse inequalities, which relate the Betti numbers $\beta_k(M) = \dim_{\mathbb{R}} H_k(M, \mathbb{R})$ to the Morse numbers of a Morse function $\mu_f(k)$ as $\beta_k(M) \leq \mu_f(k)$ and $\sum_{l=0}^{k}(-1)^l \beta_{k-l} \leq \sum_{l=0}^{k}(-1)^l \mu_f(k-l)$ for all $k \in \{0, \ldots n\}$. We follow the strategy of Witten in [8], presenting an analytic proof of these inequalities via deformations of the de Rham complex.

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# Chapter 1

# Introduction

Manifolds impart certain topological spaces with a differentiable structure. This structure can in turn tell us much about the underlying topology. One way to do this is by analysing the behaviour of functions at their critical points. To see why this is a viable strategy, consider a torus embedded in $\mathbb{R}^3$ as shown in **Figure 1** below. Consider the "height" function $f : T \to \mathbb{R}$ given by $(x, y, z) \mapsto z$ and its strict sublevel sets $M^\alpha = \{x \in M : f(x) < \alpha\}$.

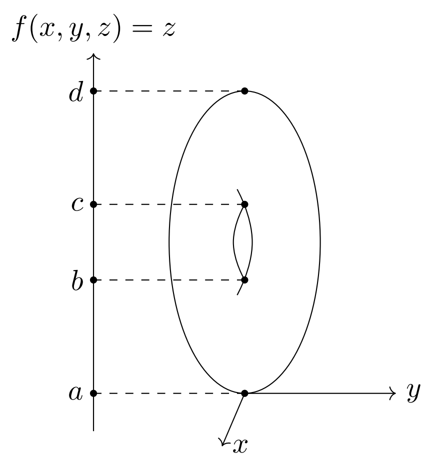



**Figure 1.** A height function on a torus.

This function has four critical points, $a, b, c, d$, as indicated. The homotopy type of the $M^\alpha$ change as $\alpha$ passes through the heights of each critical point, as shown in **Figure 2** below.

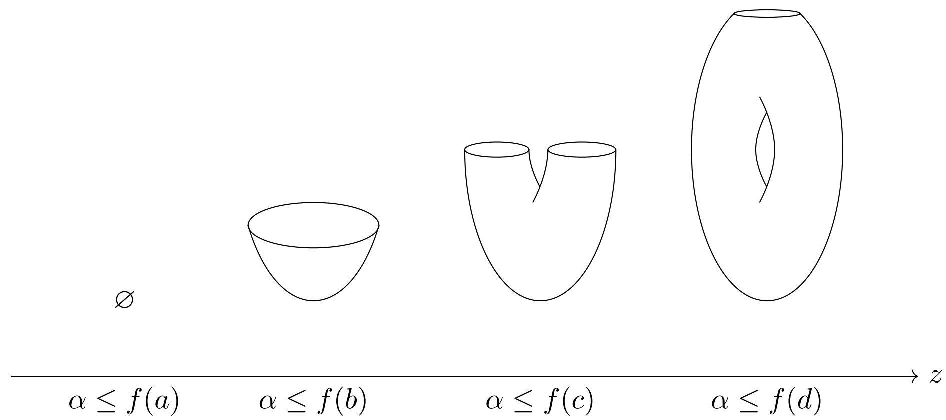



**Figure 2.** Strict sublevel sets for various heights.

The topology of the sublevel sets clearly changes at each critical point, giving us an indication as to why these functions help describe the topology - in particular, the homotopy type of the manifold. In fact, Milnor describes how passing through a critical point describes a handle attachment in [5].

We can formalise this approach with Morse Theory, a branch of differential topology. This involves finding "well-behaved" functions (i.e., those with non-degenerate critical points, such as the one above) and then describing their behaviour at its critical points. By counting the points with similar behaviours, we construct the Morse numbers $\mu_f(k)$ of the given function, which we show are intimately related to the well known Betti numbers $\beta_k(M)$ via the Morse inequalities:

**Theorem 1** (Morse Inequalities)**.** *Suppose $M$ is a closed $n$-manifold and $f : M \to \mathbb{R}$ is a Morse function. Then:*

$$\beta_k(M) \leq \mu_f(k);\ \textit{and}$$
$$\sum_{l=0}^{k}(-1)^l \beta_{k-l} \leq \sum_{l=0}^{k}(-1)^l \mu_f(k-l).$$

*for all $k \in \{0, \ldots, n\}$.*

In this thesis we present an analytic proof of these inequalities following methods outlined by Witten in [8]. In **Chapter Two** we introduce the key definitions of Morse functions as well as preliminary results about their behaviour at critical points, allowing us to define the Morse numbers $\mu_f(k)$ of a Morse function on a closed manifold. In **Chapter Three** we construct the algebra of differential forms of a manifold $\Omega(M)$, the exterior derivative $\mathrm{d}$, and the corresponding de Rham complex $(\Omega^\bullet, \mathrm{d}^\bullet)$. We explore techniques for computing the cohomology groups $H^k_{dR}(M) = \ker \mathrm{d}^k / \mathrm{im}\, \mathrm{d}^{k-1}$: the

Künneth theorem and the Mayer-Vietoris sequence. This allows us to define the Betti numbers $\beta_k(M)$ as the dimensions of the de Rham cohomology groups over $\mathbb{R}$.

In **Chapter Four** we endow our closed manifold with a Riemannian metric $g$ from which we can construct the Hodge star operator and an inner product on the space of differential forms. This defines an adjoint of the exterior derivative $\mathrm{d}^*$, known as the codifferential. Together with the exterior derivative, this defines the Dirac operator $D = \mathrm{d} + \mathrm{d}^*$ and its square, the Laplacian $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$. The Hodge decomposition theorem allows us to form a linear isomorphism from $\ker \Delta^k$ to $H^k_{dR}(M)$.

Further, we introduce the Witten deformation of the exterior derivative $\mathrm{d}_t = e^{-tf}\mathrm{d}e^{tf}$ by a Morse function $f$. Further, this gives a deformation of the codifferential $\mathrm{d}_t = e^{tf}\mathrm{d}^*e^{-tf}$, the Dirac operator $D_t = \mathrm{d}_t + \mathrm{d}_t^*$, and the Laplacian $\Delta_t = D_t^2$. We prove that this deformation does not change the zeroth eigenspace, so there is another isomorphism between $\ker \Delta$ and $\ker \Delta_t$ for all $t > 0$, allowing us to study the de Rham cohomologies via the kernel of the deformed Laplacian for large $t$.

We show that on a neighbourhood of a critical point of rank $\lambda$, the kernel of the deformed Laplacian is one dimensional and is generated by a form of rank $\lambda$. We then extend these forms to the entire manifold and use appropriate projections to form a subspace of $\Omega^k(M)$ containing the kernel of $\Delta_t$ with dimension $\mu_f(k)$ for appropriately large $t$. From this, we prove both forms of the Morse inequalities in **Chapter Five**.

Throughout this dissertation, we assume a basic knowledge of manifolds, homological algebra, and functional analysis.

# Chapter 2

# Morse Functions

We want to describe a subset on the smooth functions between manifolds called **Morse functions**. For the following, let $M$ and $N$ be (respectively) $m$ and $n$ dimensional manifolds, let $f : M \to N$ be a smooth function, and let $X, Y \in \Gamma(TM)$ be smooth vector fields on $M$.

## 2.1 Critical Points

**Definition 2.** *Let $x \in M$ and consider the differential $\mathrm{d}f_x : T_xM \to T_{f(x)}N$. We call $p$ a **critical point** of $f$ if $\mathrm{rank}(df_p) < \min(m, n)$. The set of critical points of $f$ is written $\boldsymbol{Cr}_f$.*

For our purposes we will only care about real valued functions, so the definition above immediately simplifies to the following:

**Proposition 3.** *If $N = \mathbb{R}$, then $p \in M$ is a critical point of $f$ if, and only if, $\mathrm{d}f_p = 0$.*

**Definition 4.** *We define the **directional derivative** of $f$ along $X \in \Gamma(TM)$ as $Xf : M \to \mathbb{R}$ where:*

$$(Xf)(x) = (\mathrm{d}f_x)X(x)$$

**Lemma 5.** *Let $N = \mathbb{R}$, and take some $p \in \mathbf{Cr}_f$. If $X', Y' \in \Gamma(TM)$ are vector fields on $M$ which agree with $X$ and $Y$ respectively at $p$, then we have that*

$$(XYf)(p) = (YXf)(p); \textit{ and}$$
$$(XYf)(p) = (X'Y'f)(p).$$

*Proof.* The first equality follow from the fact that $p$ is a critical point of $f$, as we see that $[U, V]f(p) = \mathrm{d}f_p[U, V](p) = 0$ (as $\mathrm{d}f_p = 0$), and so $UVf(p) = VUf(p)$ for all $U, V \in \Gamma(TM)$. Now see that $(X - X')(p) = 0$, so if $g \in C^\infty(M)$ then $(X - X')(g)(p) = \mathrm{d}g_p(X - X')(p) = 0$. Thus in particular $(X - X')(Uf)(p) = 0$, and so $XUf(p) = X'Uf(p)$ (similarly $Y'Vf(p) = YVf(p)$). So in particular, we have:

$$YXf(p) = XYf(p) = X'Yf(p) = YX'f(p) = Y'X'f(p)$$

giving us the desired equality. □

**Definition 6.** *Let $p \in \mathbf{Cr}_f$, and define the* ***Hessian*** *of $f$ at $p$ by:*

$$\begin{aligned} H_f : T_pM \times T_pM &\to \mathbb{R} \\ (X(p), Y(p)) &\mapsto (X'Y'f)(p) \end{aligned}$$

*Where $X'$ and $Y'$ are vector fields extending $X(p)$ and $Y(p)$.*

The previous lemma ensures that this is well defined; it does not depend on the choice of vector fields for $X$ and $Y$, so long as they agree with the given vectors $X(p)$ and $Y(p)$ at the critical point $p$.

**Definition 7.** *A critical point $p \in \mathbf{Cr}_f$ is called* ***non-degenerate*** *if the Hessian at $p$ is non-degenerate: $H_f(X(p), Y(p))$ vanishes for all $Y(p) \in T_pM$ if and only if $X(p) = 0$.*

## 2.2 Morse Functions

**Definition 8.** *A smooth function is called a* ***Morse function*** *if all $p \in \mathbf{Cr}_f$ are non-degenerate.*

This is a strong condition, but one that allows us to simplify the behaviour of the functions near critical points immensely. It may not be immediately obvious that such functions even exist given the strength of this condition, but in fact one can show that they are abundant in the space of continuous real-valued functions on any manifold $M$.

**Proposition 9.** *Let $M$ be a smooth manifold embedded in $\mathbb{R}^m$. For almost every point $p \in \mathbb{R}^m$, the function $f_p : M \to \mathbb{R}$ given by*

$$f_p(x) = \|x - p\|^2$$

*is a Morse function.*

**Proposition 10.** *Let $M$ be a smooth manifold embedded in $\mathbb{R}^m$, and let $f : M \to \mathbb{R}$ be a smooth function. Then $f$ and all its derivatives can be uniformly approximated by Morse functions on every compact subset of $M$.*

Both propositions are proved in [1].

## 2.3 Morse Numbers

As we will see, Morse functions are particularly nice to work with and can be exploited to provide information about the structure of the manifold. We do this by understanding the functions behaviour at each critical point and then count the critical points in a useful way to construct the Morse numbers of a given function.

**Definition 11.** *For $p \in \boldsymbol{Cr}_f$, we call the maximum dimension of a subspace of $T_pM$ on which $H_f(p)$ is negative definite the* ***Morse index*** *of $f$ at $p$. Equivalently, as $p$ is nondegenerate, the index of a critical point is the number of negative eigenvalues of the Hessian at that critical point.*

**Lemma 12** (Morse Lemma)**.** *If $f : M \to \mathbb{R}$ is a Morse function with index $\lambda$ and critical point $p$, then there exists a neighbourhood of $p$ and a local coordinate system $(x^1, \cdots, x^n)$ with $x^i(p) = 0$ for all $i$ and*

$$f(x) = f(p) - \sum_{i=1}^{\lambda}(x^i)^2 + \sum_{i=\lambda+1}^{n}(x^i)^2$$

*Proof.* Here we follow [1] and use induction on the dimension of the manifold. Consider the one-dimensional case. Taking the second order Taylor expansion for $f$ on some neighbourhood $M_p$ of $p$ (with $p$ identified with 0) gives:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \epsilon(x)x^2 = f(0) + (a + \epsilon(x))x^2$$

Define $\phi(x) = \sqrt{a + \epsilon(x))}x$, so that $f(x) = f(0) \pm \phi(x)^2$. By construction $a$ and $\epsilon$ are local diffeomorphisms, and so $\phi$ is a local diffeomorphism too. $\phi'(0) = \sqrt{a} \neq 0$ so by the inverse function theorem we can invert $\phi$, and get

$$f(x) = f \circ \phi^{-1}(x_1) = f(0) \pm (x_1)^2$$

Now consider the $n$-dimensional case. Write the coordinates as $(x, y) \in \mathbb{R} \times \mathbb{R}^n$, and write $f(x, y) = f_y(x)$, which we consider as a function of one real variable (as $y$ varies in $\mathbb{R}^n$). If $f'_y(0) = 0$, then we can proceed as in the one dimensional case and find

$$f(x, y) = f \circ \varphi^{-1}(x_1, y_1) = \pm(x_1)^2 + f(0, y_1)$$

If $f'_y(0) \neq 0$, we find a $C^\infty$ function $\phi$ with $\partial_x f(x, y) = 0$ and $x = \phi(y)$ and in the desired neighbourhood. Define $\Phi(x, y) = (x + \phi(y), y)$, so that then $g = f \circ \Phi(x, y)$ has $\partial_x g(x, y) = 0$ for all $y$ and the same Hessian as $f$. This allows us to continue as above, and then induct on the dimension of the manifold to see that

$$f(x) = f(p) + \sum_{i=1}(\pm(x^i)^2)$$

It is then clear that the Hessian of $f$ at $p$ is represented by the matrix with $\pm 2$ on the diagonal and 0 elsewhere. Thus there are exactly $\lambda$ negative terms, and so by permuting the $x^i$ we get the desired expression for $f(x)$. □

We often call such coordinates (or rather, an appropriate scaling of these coordinates by a half so that $|df_u| = |u|$) a **Morse chart** corresponding to $f$. These charts are particularly useful as they allow us to write:

$$\mathrm{d}f = -x^1 \mathrm{d}x^1 - \cdots - x^\lambda \mathrm{d}x^\lambda + x^{\lambda+1}\mathrm{d}x^{\lambda+1} + \cdots + x^n \mathrm{d}x^n$$

for all $x$ in the neighbourhood of $p \in \mathbf{Cr}_f$ covered by the chart.

**Corollary 13.** *The critical points of a Morse function are isolated.*

*Proof.* The expansion of $f$ on this neighbourhood of a critical point has only one critical point. □

**Corollary 14.** *The number of critical points of a Morse function on a closed manifold is finite.*

This means that we can count the critical points of a given Morse function in a useful way, allowing us to define one side of the Morse inequalities.

**Definition 15.** *Let $f$ be a Morse function with finitely many critical points $p \in \boldsymbol{Cr}_f$ with indices $\lambda_p$ on the compact, smooth $n$-manifold $M$. We define the **Morse polynomial** of $f$ to be*

$$P_f(x) = \sum_{p \in \boldsymbol{Cr}_f} x^{\lambda_p} = \sum_{l=0}^{n} \mu_f(l) x^l$$

*where $\mu_f(l) = |\{p \in \boldsymbol{Cr}_f : \lambda_p = l\}|$ is the number of critical points with index $l$, which we call the **Morse numbers** of $f$.*

**Example 16.** *Consider the 2-torus $T^2$, and let $f$ be the height function as defined in **Chapter 1**. We see that the lowest critical point, $a$, is a minimum, and so has index 0. Similarly, the highest critical point $d$ is a maximum and so has index 2. The middle two critical points $b$ and $c$ are both saddles with index 1. Thus the Morse numbers of $f$ are $\mu_f(0) = 1$, $\mu_f(1) = 2$, $\mu_f(2) = 1$, and $\mu_f(k) = 0$ for all other $k$.*

# Chapter 3

# De Rham Cohomology

In this chapter we define the Betti numbers of a closed orientable manifold $M$. We do this by showing that the exterior derivative turns spaces of differetial forms into a cochain complex. The dimensions of the cohomology groups of this complex are the Betti numbers. We also show that these are readily computable by providing techniques and examples in **Section 3.3**.

## 3.1 Differential Forms and the Exterior Derivative

**Definition 17.** *We define $\Omega^k(M) = \Lambda^k(T^*M)$ as the space of completely skew $k$-forms on $M$, and define the space of differential forms $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$, a $C^\infty(M)$ module endowed with the bilinear (graded) commutative **wedge** product $\wedge : \Omega(M) \times \Omega(M) \to \Omega(M)$.*

**Remark 18.** *A simple counting argument shows that if $M$ is an $n$-dimensional manifold then $\Omega^k(M)$ can be generated over $C^\infty(M)$ by $\binom{n}{k}$ forms $\{\mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k} : 1 \leq i_1 < \cdots i_k \leq n\}$, and that $\Omega^n(M) = 0$ for $n > \dim M$.*

**Definition 19.** *Let $X, X_i \in \Gamma(TM)$ for each $i \in \mathbb{N}$. Define the **interior product** with $X$ as the map $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ defined by:*

$$(\iota_X \omega)(X_1, \ldots, X_{k-1}) = \omega(X, X_1, \ldots, X_{k-1})$$

*We often write $\iota_X(\omega)$ as $X \lrcorner\, \omega$ and call it the **contraction** of $X$ and $\omega$.*

**Proposition 20.** *The interior derivative is $C^\infty$-linear and for $\omega \in \Omega^k(p)$ and $\eta \in \Omega^j(M)$ satisfies the graded Liebniz rule:*

$$X \lrcorner (\omega \wedge \eta) = (X \lrcorner\, \omega) \wedge \eta + (-1)^k \omega \wedge (X \lrcorner \eta)$$

For a proof, see [6].

**Definition 21.** *Let $M$ be a smooth manifold, $X_1, \ldots, X_{k+1} \in \Gamma(TM)$ be vector fields on $M$, and $\Omega^k(M)$ be the set of completely skew $k$-forms on $M$. We define the* ***exterior derivative*** $\mathrm{d}^k : \Omega^k(M) \to \Omega^{k+1}(M)$ *as:*

$$\begin{aligned}\mathrm{d}^k\omega(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^i \, X_i \, \omega(X_1, \ldots, \widehat{X_i}, \ldots X_{k+1}) \\ &+ \sum_{i=1}^{k+1}\sum_{j=1}^{i}(-1)^{i+j}\omega([X_j, X_i], V_1, \ldots, \widehat{X_j}, \ldots, \widehat{X_i}, \ldots X_{k+1})\end{aligned}$$

*Where $[X_i, X_j]$ is the Lie bracket of $X_i$ and $X_j$, and $\widehat{X_i}$ mean omitting $X_i$.*

The fact that $\mathrm{d}^k\omega$ (as defined above) is in fact a completely skew $k+1$ tensor on $T^*M$ follows immediately from the properties of $\omega$ and the Lie bracket. We call forms for which $\mathrm{d}\omega = 0$ **closed** and forms $\omega$ for which $\omega = \mathrm{d}\alpha$ for some $\alpha$ **exact**. The exterior derivative has several useful properties.

**Proposition 22.** *The exterior derivative satisfies the following properties:*

1. $\mathrm{d}$ *is $\mathbb{R}$-linear;*
2. $\mathrm{d}$ *commutes with pullbacks, i.e., $f^*\mathrm{d}(\omega) = \mathrm{d}(f^*\omega)$;*
3. $\mathrm{d}(\mathrm{d}f) = 0$ *for all $f \in C^\infty(M)$; and*
4. $\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k(\alpha \wedge (\mathrm{d}\beta))$ *for any $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^j(M)$.*

*Moreover, any graded derivation $\delta : \Omega^k(M) \to \Omega^{k+1}(M)$ that satisfies these properties agrees with the exterior derivative.*

For a proof, see [6], where it is also shown that there is a convenient local expression for d:

**Proposition 23.** *If $\omega \in \Omega^k(M)$ is given in local coordinates on some chart $U$ by*

$$\omega = \omega_{i_1 \cdots i_k} \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k}$$

*then the exterior derivative is given locally by:*

$$\mathrm{d}^k\omega = \partial_j \omega_{i_1 \cdots i_k} \mathrm{d}u^j \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k}$$

**Proposition 24.** *Suppose $M$ is endowed with a Riemannian metric $g$, and that in a local chart $U \subseteq M$ the coordinate basis $\partial_i$ form an orthonormal frame for $TM$ with dual basis $\mathrm{d}u^i \in T^*M$, then*

$$\mathrm{d} = \sum_{i=1}^{n} \mathrm{d}u^i \wedge \nabla_{\partial_i}.$$

*Proof.* First, suppose that $(\overline{\partial}_1, \ldots, \overline{\partial}_n)$ is another orthonormal frame on $U$, with $\overline{\partial}_i = a_i{}^k \partial_k$ and $\mathrm{d}v^i = b^i{}_k \mathrm{d}u^k$. Thus we have that

$$\mathrm{d}v^i \wedge \nabla_{\overline{\partial}_i} = (b^i{}_k \mathrm{d}u^k) \wedge \nabla_{a_i{}^k \partial_k} = b^i{}_k a_i{}^k \mathrm{d}u^k \wedge \nabla_{\partial_k} = \mathrm{d}u^k \wedge \nabla_{\partial_k}$$

as we have $a_i{}^k b^i{}_l = \delta^k{}_l$. Thus the right hand side is independent of choice of coordinates. Therefore, we can work in normal coordinates, for which the connection simplifies to $\nabla_{\partial_i} = \partial_i$ at a given point $u$. From this it is clear that it agrees with the local form in **Proposition 23** by the $C^\infty$ linearity of $\wedge$. Then again by the coordinate independence, this extends everywhere. □

**Example 25.** *For a smooth function $f \in C^\infty(M) = \Omega^0(M)$, its exterior derivative is just its differential:*

$$\mathrm{d}^0 f(V)(x) = -Vf(x) = (\mathrm{d}f)V(x).$$

*For a 1-form $\omega$, its exterior derivative is:*

$$\mathrm{d}^1\omega(V_1, V_2) = V_1\omega(V_2) - V_2\omega(V_1) - \omega([V_1, V_2]).$$

## 3.2 The de Rham Complex

**Proposition 26.** *The composition $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$ for all $k \geq 0$.*

*Proof.* Take some chart $U \subseteq M$ and form $\omega \in \Omega^k(M)$, and express it in coordinates on $U$ as $\omega = \omega_{i_1 \cdots i_k}(\mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k})$, where each $\omega_{i_k \cdots i_k} \in C^\infty(M) = \Omega^0(M)$. Then we have

$$\begin{aligned}
\mathrm{d}^{k+1} \circ \mathrm{d}^k \omega &= \mathrm{d}^{k+1} \circ \mathrm{d}^k(\omega_{i_1 \cdots i_k}(\mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k})) \\
&= (\partial_j \partial_i \omega_{i_1 \cdots i_k})(\mathrm{d}u^j \wedge \mathrm{d}u^i \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k}) \\
&= -(\partial_i \partial_j \omega_{i_1 \cdots i_k})(\mathrm{d}u^i \wedge \mathrm{d}u^j \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \mathrm{d}u^{i_k}) \\
&= -\mathrm{d}^{k+1} \circ \mathrm{d}^k \omega
\end{aligned}$$

because the partials commute and the wedge product is skew. Thus indeed $d \circ d = 0$ on $U$. Since d commutes with pullbacks **Proposition 22**, it is invariant under coordinate transforms, and so $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$ globally. □

**Corollary 27.** *The sequence* $(\Omega^k(M), \mathrm{d}^k)_{k\geq 0}$, *is a cochain complex.*

We often drop the $k$ from $\mathrm{d}^k$ and rely on the context of the form it is acting on for this information.

**Definition 28.** *The* $k$*-th* ***de Rham cohomology group*** $H^k_{dR}(M)$ *is the* $k$*-th cohomology groups of the cochain* $(\Omega^\bullet(M), \mathrm{d}^\bullet)$*;*

$$H^k_{dR}(M) = \frac{\ker \mathrm{d}^k}{\operatorname{im} \mathrm{d}^{k-1}}$$

These de Rham cohomology groups are readily computable as we show in **Section 3.3**. More importantly, by de Rham's theorem the de Rham cohomology groups are isomorphic to the singular cohomology of the manifold with real coefficients $H^k(M, \mathbb{R})$.

**Theorem 29** (De Rham's Theorem)**.** *Define the map* $I : H^k_{dR}(M) \to H^k(M, \mathbb{R})$*, for which if* $[\omega] \in H^k_{dR}(M)$*, then* $I([\omega])$ *is the element of* $Hom(H_k(M, \mathbb{R}), \mathbb{R}) \simeq H^k(M, \mathbb{R})$ *that acts as:*

$$[c] \mapsto \int_c \omega$$

*for smooth* $c$*. Then* $I$ *is an isomorphism, and thus* $H^k_{dR}(M) \simeq H^k(M, \mathbb{R})$*.*

This means that the de Rham cohomology groups are in fact homotopy invariants (i.e., invariants under continuous deformations) of the manifold. We sometimes drop the subscripted $dR$ because of this. These cohomology groups capture much of the topological information of the manifold, especially the well known Betti numbers.

**Definition 30.** *We define the* ***Betti numbers*** $\beta_k$ *of a smooth manifold* $M$ *to be the dimensions of the de Rham cohomology groups*

$$\beta_k(M) = \dim_{\mathbb{R}} H^k_{dR}(M)$$

**Proposition 31.** *Let* $M$ *be a smooth* $n$*-manifold, then* $\beta_k(M) = 0$ *if* $k < 0$ *or* $k > n$*.*

*Proof.* This follows from the fact that $\dim_{C^\infty(M)} \Omega^k(M) = \binom{n}{k}$ for $0 \leq k \leq n$ and 0 otherwise. □

To show the importance of these numbers, we note that they are used to define the well known Euler characteristic.

**Definition 32.** *The* ***Poincaré polynomial*** *of a smooth* $n$*-manifold* $M$ *is defined as the generating function of the Betti numbers,*

$$P_M(t) = \sum_{k=0}^{n} \beta_k(M) t^k$$

**Definition 33.** *The **Euler characteristic** of a smooth n-manifold is*

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \beta_k(M) = P_M(-1)$$

## 3.3 Computations of Cohomology Groups

In order to calculate the de Rham cohomology groups and the consequent Betti numbers, we utilise well known tools from homological algebra; namely the Künneth theorem and the Zigzag lemma. This allows us to construct cohomology groups for manifolds by analysing smaller and more workable examples first. We first consider certain subsets of $\mathbb{R}^n$, then the circle $S^1$ and n-tori.

**Theorem 34** (Poincaré Lemma)**.** *Suppose that $U \subseteq \mathbb{R}^n$ is a star like set, i.e., an open set for which there exists a point $x_0 \in U$ with $\{(1-t)x_0 + tx : t \in [0,1]\} \subseteq U$ for all $x \in U$. Then:*

$$H^k_{dR}(U) = \begin{cases} \mathbb{R}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}$$

The proof is given in [2]. It is clear that $\mathbb{R}^n$ is itself star-like, which gives us:

**Corollary 35.** *$H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$ and $H^k_{dR}(\mathbb{R}^n) = 0$ for all other $k$.*

We now shift focus to a general closed manifold $M$.

**Proposition 36.** *Let $M$ be a closed manifold. Then $H^0(M) \simeq \mathbb{R}^a$ where $a = \dim H^0(M)$ is the number of connected components of $M$.*

*Proof.* From **Example 25**, we see that $[f] \in H^0_{dR}(M)$ if and only if $\mathrm{d}f = 0$, which implies that $f$ is constant on each connected component of $M$. Thus $H^0(M) = \ker \mathrm{d}/\mathrm{d}(\{0\}) \simeq \ker \mathrm{d} \simeq \mathbb{R}^a$. □

Consider a smooth manifold $M$ covered by open sets $U$ and $V$. Let $\iota_{A,B} : A \to B$ be the inclusion map of $A$ in $B$. The maps $\iota_{U,M}$, $\iota_{V,M}$, $\iota_{U\cap V,U}$, and $\iota_{U\cap V,V}$ induce contravariant inclusion maps on the forms on these sets: $\iota^*_{U,M} : \Omega^\bullet(M) \to \Omega^\bullet(U)$ given by $\omega \mapsto \omega|_U$ etc.. Also define $k^* = \iota^*_{U,M} \oplus \iota^*_{V,M} : \Omega^\bullet(M) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V)$ by $\omega \mapsto (\iota^*_{U,M}\omega, \iota^*_{V,M}\omega)$, and $j^* = \iota^*_{U\cap V,U} - \iota^*_{U\cap V,V} : \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V)$ by $(\omega, \eta) \mapsto \iota^*_{U\cap V,U}\omega - \iota^*_{U\cap V,V}\eta$. These six induced maps are all linear, and they all commute with the exterior derivative (as they are all just restrictions), thus they induce maps on the cohomology groups of these modules (we denote these by the same names).

**Theorem 37** (Mayer-Vietoris sequence)**.** *If a smooth manifold $M$ is covered by open sets $U$ and $V$, then for each $n$ the following sequence is exact:*

$$0 \longrightarrow \Omega^n(M) \xrightarrow{k^*} \Omega^n(U) \oplus \Omega^n(V) \xrightarrow{j^*} \Omega^n(U \cap V) \longrightarrow 0$$

This then allows us to apply the Zigzag lemma (proved in **Appendix A**) to the short exact sequence of cochain complexes above to construct a connecting homomorphism $\delta$.

**Corollary 38.** *The following is a long exact sequence.*

$$\cdots \longrightarrow H^n(M) \xrightarrow{k^*} H^n(U) \oplus H^n(V) \xrightarrow{j^*} \Omega^n(U \cap V) \xrightarrow{\delta} H^{n+1}(M) \longrightarrow \cdots$$

This sequence, also known as the Mayer-Vietoris sequence gives us a powerful tool for computing de Rham cohomology groups. In particular, if we can cover our manifold by two open sets with the same homotopy type as $\mathbb{R}^n$, then we can use the Poincaré lemma and the Mayer-Vietoris sequence to find the cohomology groups of the manifold.

**Example 39.** *Consider the circle $S^1$, covered by two open sets overlapping as in the figure below.*

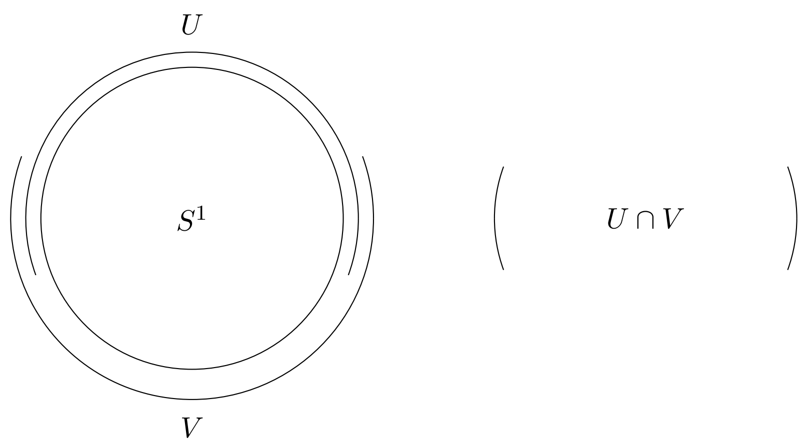



***Figure 3.*** *A chart on a circle.*

*Both $U$ and $V$ have the same homotopy type as $\mathbb{R}$, and $U \cap V$ has two components each with the homotopy type of $\mathbb{R}$. Thus $H^0(U) \simeq H^0(V) \simeq \mathbb{R}$ and $H^0(U \cap V) \simeq \mathbb{R} \oplus \mathbb{R}$, and $H^n(U) \simeq H^n(V) \simeq H^n(U \cap V) \simeq 0$. Thus the Mayer-Vietoris sequence for the circle is:*

$$0 \longrightarrow H^0(S^1) \xrightarrow{k^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\delta} H^1(S^1) \longrightarrow 0$$

*Hence $\delta$ is surjective and so $H^1(S^1) \simeq \mathrm{Im}\ \delta = \ker j^* \simeq \mathbb{R}$ (the last isomorphism here follows from the fact that the subtraction map $j^*$ clearly has kernel $\{(\omega, \omega) : \omega \in H^0(U)\}$). By **Proposition 36**, we also know that $H^0(S^1) \simeq \mathbb{R}$, and so $H^0(S^1) \simeq H^1(S^1) \simeq \mathbb{R}$ whilst $H^n(S^1) = 0$ for all other $n$.*

These calculations for "small" manifolds quickly generalise to higher dimensional product manifolds via the well known Künneth theorem of homological algebra.

**Theorem 40** (Künneth Theorem)**.** $H^k(X \times Y) \simeq \bigoplus_{i+j=k} H^i(X) \otimes H^j(Y)$

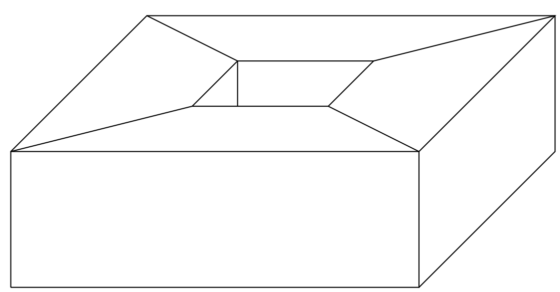
**Corollary 41.** *If $p_X(x)$ and $p_Y(x)$ are generating functions for the Betti numbers of $X$ and $Y$ (respectively), then $p_X(x)p_Y(x)$ is the generating function of the Betti numbers of $X \times Y$.*

Since we are primarily interested in the Betti numbers of a manifold, this gives us a powerful tool for computing them. Since we have constructed the Betti numbers of $\mathbb{R}^n$ using the Poincaré lemma and of the circle $S^1$ via the Mayer-Vietoris sequence, we can use their generating functions to find the Betti numbers of any manifold $M$ homotopically equivalent to a direct product of these spaces.

**Example 42.** *As seen previously in **Example 39**, the circle $S^1$ has Betti numbers $1, 1$ and thus $p_{S^1}(x) = 1 + x$. The $n$-torus $T^n$ is isomorphic to the product $(S^1)^n$, thus $p_{T^n}(x) = (1+x)^n$, and hence the Betti numbers of the $n$-torus are given by the $n^{th}$ row of Pascal's triangle. That is,*

$$H^k_{dR}(T^n) \simeq \mathbb{R}^{\binom{n}{k}}.$$

*This implies, for instance, that the Betti numbers of the 2-torus are 1,2,1, and so its Euler characteristic is $\chi(T^2) = 1 - 2 + 1 = 0$ which agrees with the genus definition $\chi(T^2) = 2 - 2g = 0$ and a homotopic polyhedral approximation:*



*which has $16$ vertices, $32$ edges, and $16$ faces and thus has an Euler characteristic of $V - E + F = 16 - 32 + 16 = 0$. We also note that these Betti numbers are equal to the Morse numbers of the height function described in **Example 16**.*

# Chapter 4

# Hodge Theory and Witten Deformations

## 4.1 Inner Product on the Space of Differential Forms

Take a closed $n$-manifold $M$ equipped with a Riemannian metric $g$. This induces a pairing on contravariant $k$-tensors $S = S_I \mathrm{d}u^I, T = T_J \mathrm{d}u^J \in \Gamma(T^*M^{\oplus k})$ given in local coordinates by

$$\langle S, T \rangle_g = g^{a_1 b_1} \cdots g^{a_k b_k} S_{a_1 a_2 \cdots a_k} T_{b_1 b_2 \cdots b_k}$$

The metric also induces a canonical volume form $\mathrm{d}M \in \Omega^n(M)$, given in local coordinates by $\mathrm{d}M = \sqrt{|g|}\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^n$. Since $\wedge$ is $C^\infty$ linear, for any $\eta \in \Omega^k(M)$ and $h \in C^\infty(M)$ it is clear that we can always find a form $\omega \in \Omega^{n-k}(M)$ for which $\eta \wedge \omega = h\mathrm{d}M$ and that this form must be unique.

**Definition 43.** *Fix $k \in \{1, \ldots, n-1\}$. Define the* ***Hodge star operator*** $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ *of $\omega \in \Omega^k(M)$ by the unique $n-k$ form that satisfies*

$$\eta \wedge (\star\omega) = \langle \eta, \omega \rangle_g \mathrm{d}M$$

*for all $\eta \in \Omega^k(M)$.*

The following is then immediately clear from the definition:

**Proposition 44.** $\star \star \omega = (-1)^{k(n-k)}\omega$*, and thus*

$$\star^{-1} = \begin{cases} \star & \text{if } n \text{ is odd} \\ (-1)^k \star & \text{if } n \text{ is even} \end{cases}$$

**Definition 45.** *We define an inner product on each* $\Omega^k(M)$ *by*

$$\langle \omega, \eta \rangle = \int_M \omega \wedge \star\eta = \int_M \langle \omega, \eta \rangle_g \mathrm{d}M$$

For a proof that this is an inner product see [7].

## 4.2 The Laplacian and Harmonic Forms

The Hodge star operator allows us to define the adjoint of the exterior derivate, explicitly:

**Definition 46.** *The* ***codifferential*** *is given by* $\mathrm{d}^* : \Omega^k(M) \to \Omega^{k-1}(M)$ *by* $\mathrm{d}^* = (-1)^{nk+n+1} \star \mathrm{d}\star = (-1)^k \star^{-1} \mathrm{d}\star$.

**Proposition 47.** $\mathrm{d}^*$ *is the adjoint to the exterior derivative* $\mathrm{d}$ *with respect to the inner product* $\langle \cdot, \cdot \rangle$*, i.e.,* $\langle \mathrm{d}\omega, \eta \rangle = \langle \omega, \mathrm{d}^*\eta \rangle$ *for all* $\omega \in \Omega^{k-1}(M)$ *and* $\eta \in \Omega^k(M)$*.*

*Proof.* Using Stokes' theorem, and noting that $M$ is closed and so $\partial M = \varnothing$, we see that:

$$\begin{aligned}
\langle \mathrm{d}\omega, \eta \rangle - \langle \omega, \mathrm{d}^*\eta \rangle &= \int_M (\mathrm{d}\omega) \wedge \star\eta - \int_M \omega \wedge \star(\mathrm{d}^*\eta) \\
&= \int_M \mathrm{d}(\omega \wedge \star\eta) + (-1)^k \omega \wedge \mathrm{d}(\star\eta) - \int_M \omega \wedge \star(-1)^k \star^{-1} \mathrm{d} \star \eta \\
&= \int_{\partial M} \langle \omega, \eta \rangle_g \mathrm{d}M + \int_M (-1)^k \omega \wedge \mathrm{d}(\star\eta) - \int_M \omega \wedge (-1)^k \mathrm{d}(\star\eta) \\
&= 0
\end{aligned}$$

So indeed $\langle \mathrm{d}\omega, \eta \rangle = \langle \omega, \mathrm{d}^*\eta \rangle$. □

**Lemma 48.** $\mathrm{d}^* \circ \mathrm{d}^* = 0$.

*Proof.* We have $\langle \eta, \mathrm{d}^* \circ \mathrm{d}^*\omega \rangle = \langle \mathrm{d} \circ \mathrm{d}\eta, \omega \rangle = 0$ for all $\eta \in \Omega^k(M)$ and $\omega \in \Omega^{k+2}$, hence $\mathrm{d}^* \circ \mathrm{d}^* = 0$. □

We can also find a local description of the codifferential in terms of the Levi-Civita connection, similarly to the exterior derivative in **Propositition 24**.

**Proposition 49.** *Suppose* $M$ *is endowed with a Riemannian metric* $g$*,and that on a local chart* $U \subseteq M$ *the coordinate basis* $\partial_i$ *form an orthonormal frame for* $TM$ *with dual basis* $\mathrm{d}u_i \in T^*M$*, then*

$$\mathrm{d}^* = -\sum_{i=1}^n \partial_i \lrcorner \nabla_{\partial_i}.$$

*Proof.* Analogously to **Proposition 24**, take another orthonormal frame $(\overline{\partial}_1, \dots, \overline{\partial}_n)$ for $TM$ with dual frame $(\mathrm{d}v^1, \dots, \mathrm{d}v^n)$. Expand as before and see that

$$\sum_{i=1}^{n} \overline{\partial}_i \lrcorner \nabla_{\overline{\partial}_i} = {a_i}^k b_i^k \partial_i \lrcorner \nabla_{\partial_i} = \partial_i \lrcorner \nabla_{\partial_i}$$

as ${a_i}^k {b_i}^l = \delta^{kl}$. So indeed both sides are independent of coordinates.

Suppose that $(u^1, \dots, u^n)$ form normal coordinates on $U$, so that the induced frame $(\partial_i, \dots, \partial_n)$ is orthonormal at its center $x_0$. Take $I = (i_1, \dots, i_k)$, and define $\varepsilon : \{0, \dots, n\} \to \{0, \dots, k\}$ by $i_a \mapsto a$. By the $C^\infty$-linearity and the Leibniz rule of the interior product:

$$\begin{aligned} \partial_l \lrcorner \mathrm{d}u^I &= \sum_{j=1}^{k} (-1)^{\varepsilon(j)+1} \delta_{l i_j} \mathrm{d}u^{i_1} \wedge \dots \wedge \widehat{\mathrm{d}u^j} \wedge \dots \wedge \mathrm{d}u^{i_k} \\ &= (-1)^{\varepsilon(l)+1} \mathrm{d}u^{i_1} \wedge \dots \wedge \widehat{\mathrm{d}u^l} \wedge \dots \wedge \mathrm{d}u^{i_k} \end{aligned}$$

if $l = i_a$ for some $a$, and 0 otherwise (where $\varepsilon(l) = a$). Then see that

$$\begin{aligned} \star \partial_l \lrcorner \mathrm{d}u^I &= \star (-1)^{a+1} \mathrm{d}u^{i_1} \wedge \dots \wedge \widehat{\mathrm{d}u^{i_a}} \wedge \dots \wedge \mathrm{d}u^{i_k} \\ &= (-1)^{a+1} \epsilon_1 \mathrm{d}u^{i_a} \wedge \mathrm{d}u^{j_1} \wedge \dots \wedge \mathrm{d}u^{j_{n-k}} \\ &= (-1)^{a+1} \epsilon_1 \mathrm{d}u^{i_a} \wedge \epsilon_2 \star \mathrm{d}u^I \end{aligned}$$

Where $\epsilon_1$ is 1 if $(i_1, \dots, \widehat{i_a}, \dots, i_k, i_a, j_1, \dots, j_{n-k})$ is an even permutation of $(1, \dots, n)$ and $-1$ otherwise. Similarly, $\epsilon_2$ is 1 if $(i_1, \dots, i_k, j_1, \dots, j_{n-k})$ is an even permutation of $(1, \dots, n)$ and $-1$ otherwise. Hence $\epsilon_2 = (-1)^{k-a} \epsilon_1$ and so

$$\partial_l \lrcorner \mathrm{d}u^I = (-1)^{k+1} \star^{-1} (\mathrm{d}u^{i_a} \wedge \star \mathrm{d}u^I).$$

Now take a $k$-form $\omega = \omega_{i_1 \dots i_k} \mathrm{d}u^{i_1} \wedge \dots \wedge \mathrm{d}u^{i_k} = \omega_I \mathrm{d}u^I$. We have:

$$\begin{aligned} \mathrm{d}^* \omega &= (-1)^k \star^{-1} \mathrm{d} \star (\omega_I \mathrm{d}u^I) = (-1)^k \star^{-1} \mathrm{d}(\omega_I \star (\mathrm{d}u^I)) \\ &= (-1)^k \star^{-1} (\partial_l \omega_I) \mathrm{d}u^l \wedge \star(\mathrm{d}u^I)) \\ &= (-1)(\partial_l \omega_I)(-1)^{k+1} \star^{-1} (\mathrm{d}u^l \wedge \star(\mathrm{d}u^I)) \\ &= (-1)(\partial_l \omega_I) \partial_l \lrcorner \mathrm{d}u^I = -\sum_{l=1}^{n} \partial_l \lrcorner \nabla_{\partial_l} \omega \end{aligned}$$

in this coordinate system. By the independence shown above we then get the desired result. □

**Definition 50.** *The **Dirac operator** $D : \Omega(M) \to \Omega(M)$ is defined by the sum $D = \mathrm{d} + \mathrm{d}^*$. The **Laplacian** of the de Rham complex is the linear operator*

$$\begin{aligned} \Delta^k : \Omega^k(M) &\to \Omega^k(M) \\ \omega &\mapsto \mathrm{d}\mathrm{d}^*\omega + \mathrm{d}^*\mathrm{d}\omega = D^2\omega. \end{aligned}$$

This is a generalisation of the multivariable calculus Laplacian $\Delta = \nabla \cdot \nabla f$ from functions to differential forms, as $\mathrm{d}^* f = 0$ and so $\Delta^0 f = \mathrm{d}^*\mathrm{d}f = (-1)^0 \star \mathrm{d} \star \mathrm{d}f = \star \mathrm{d} \star ((\mathrm{d}f)^\sharp)^\flat = -\mathrm{div}\ \mathrm{grad} f$. In fact:

**Proposition 51.** *On a local chart $U \subseteq M$, if $\partial_i$ form an orthonormal frame for $TM$, then*

$$\Delta\omega = -\sum_{i=1}^n (\partial_i)^2\omega.$$

*Proof.* Note that $\mathrm{d} + \mathrm{d}^* = \sum_{i=1}^n c(\partial_i)\nabla_{\partial_i}$ where $c(\partial_i) = (\mathrm{d}u^i \wedge) - (\partial_i \lrcorner)$. Since $c$ is a Clifford map (see **Appendix B**), by the commutativity of $\nabla_{\partial_i}$ and $\nabla_{\partial_j}$ on this chart we have:

$$\begin{aligned} (\mathrm{d} + \mathrm{d}^*)^2 = \left(\sum_{i=1}^n c(\partial_i)\nabla_{\partial_i}\right)^2 &= \sum_{i,j=1}^n c(\partial_i)\nabla_{\partial_i} c(\partial_j)\nabla_{\partial_j} \\ &= \sum_{i,j=1}^n c(\partial_i)(c(\nabla_{\partial_i}\partial_j) + c(\partial_j)\nabla_{\partial_i})\nabla_{\partial_j} = \sum_{i,j=1}^n c(\partial_i)c(\partial_j)\nabla_{\partial_i}\nabla_{\partial_j} \\ &= \sum_{i=1}^n c(\partial_i)^2\nabla_{\partial_i}^2 + \sum_{i=1}^n\sum_{j=1}^{i-1}(c(\partial_i)c(\partial_j) + c(\partial_j)c(\partial_i))\nabla_{\partial_i}\nabla_{\partial_j} \end{aligned}$$

Utilising the Clifford relations in **Appendix B**, we have $c(\partial_i)^2 = -1$ and $c(\partial_i)c(\partial_j) + c(\partial_j)c(\partial_i) = 0$ for $i \neq j$, which gives the required equality. □

**Definition 52.** *The **harmonic $k$-forms** on a manifold $M$ are the forms $\alpha \in \Omega^k(M)$ such that $\Delta^k\alpha = 0$. We write:*

$$\mathcal{H}^k_\Delta(M) = \{\alpha \in \Omega^k(M) : \Delta^k\alpha = 0\} = \ker \Delta^k$$

**Lemma 53.** *A form $\omega$ is harmonic if and only if $\mathrm{d}\omega = \mathrm{d}^*\omega = 0$.*

*Proof.* Note that if $\omega \in \ker \Delta$ then by the properties of the inner product $0 = \langle \Delta\omega, \omega\rangle = \langle \mathrm{d}\mathrm{d}^*\omega, \omega\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \omega\rangle = \langle \mathrm{d}^*\omega, \mathrm{d}^*\omega\rangle + \langle \mathrm{d}\omega, \mathrm{d}\omega\rangle \geq 0$, so we must have $\mathrm{d}\omega = \mathrm{d}^*\omega = 0$. The converse is clear. □

**Proposition 54.** *The operator $\Delta$ is self-adjoint.*

*Proof.* Note that $\langle \Delta\omega, \eta\rangle = \langle \mathrm{dd}^*\omega, \eta\rangle + \langle \mathrm{d}^*\mathrm{d}\omega, \eta\rangle = \langle \omega, \mathrm{dd}^*\eta\rangle + \langle \omega, \mathrm{d}^*\mathrm{d}\eta\rangle = \langle \omega, \Delta\eta\rangle$. □

This implies (in particular) that harmonic forms are closed $\mathcal{H}^k_\Delta(M) \subseteq \ker \mathrm{d}^k$, and so the canonical homomorphism from $\ker \mathrm{d}^k$ to $H^k(M)$ restricted to $\mathcal{H}^k_\Delta(M)$ is a well-defined homomorphism. We will show that this is in fact an isomorphism, using the Hodge decomposition theorem for self-adjoint elliptic differential operators. The proof of this theorem can be found in [7].

**Theorem 55** (Hodge Decomposition Theorem for $\Delta$)**.** *Let $(M, g)$ be a closed, oriented Reimannian manifold. Then $\Omega^k(M)$ admits an orthogonal decomposition:*

$$\Omega^k(M) = \ker \Delta_k \oplus \operatorname{im} \Delta^k(M) = \mathcal{H}^k_\Delta(M) \oplus \mathrm{d}(\Omega^{k-1}(M)) \oplus \operatorname{im} \mathrm{d}^*(\Omega^{k+1}(M))$$

*Moreover,* $\ker \Delta^k$ *is finite dimensional over* $\mathbb{R}$.

**Theorem 56.** *Let $M$ be a closed Riemannian manifold. The homomorphism $\phi$ : $\mathcal{H}^k_\Delta(M) \to \mathcal{H}^k_\Delta(M)/\mathrm{Im}\ \mathrm{d} \subseteq H^k_{dR}(M)$ is in fact an isomorphism.*

*Proof.* Suppose that $\mathcal{H}^k_\Delta(M) \ni \gamma, \eta \mapsto [\omega] \in H^k_{dR}(M)$, i.e., they are cohomologous. Then $\gamma = \eta + \mathrm{d}\alpha$ for some $\mathrm{d}\alpha \in \mathrm{Im}\ \mathrm{d}$. Then $\mathrm{d}\alpha = \gamma - \eta \in \mathcal{H}^k_\Delta(M)$, and so by **Lemma 53** $\mathrm{d}^*\mathrm{d}\alpha = 0$. Thus $0 = \langle \alpha, \mathrm{d}^*\mathrm{d}\alpha\rangle = \langle \mathrm{d}\alpha, \mathrm{d}\alpha\rangle$ and so $\mathrm{d}\alpha = 0$ and $\eta = \gamma$, giving us injectivity. Now take $\omega \in \ker \mathrm{d}^k$, and expand as $\omega = \mathrm{d}\alpha + \mathrm{d}^*\beta + \gamma$. Note that $\mathrm{d}(\omega - \mathrm{d}\alpha) = 0$, and that $\mathrm{d}^*(\omega - \mathrm{d}\alpha) = \mathrm{d}^*(\mathrm{d}^*\beta + \gamma) = 0$. Thus (again by **Lemma 53**) $\mathcal{H}^k_\Delta(M) \ni \omega - \mathrm{d}\alpha \mapsto [\omega]$ which proves surjectivity. □

Importantly, this means that every cohomology class in $H^k_{dR}(M)$ contains a unique harmonic form. This description of the de Rham cohomology groups will be essential for proving the Morse inequalities. We also get two important corollaries:

**Corollary 57.** *If $M$ is a closed $n$-manifold, then $H^k_{dR}(M)$ is finite dimensional for each $k$.*

**Corollary 58.** *Since $H^k_{dR}(M)$ is independent of the metric, so is* $\ker \Delta^k$ *for every* $k \geq 0$.

This independence is important, since we may now choose a metric so that on each Morse chart around a critical point, the coordinate basis $\partial_i$ are orthonormal.

## 4.3 Witten Deformations

Fix a Morse function, $f$, on a closed $n$-manifold $M$. Take $t \in \mathbb{R}$ and define:

$$\mathrm{d}_t = e^{-tf}\mathrm{d}e^{tf},$$

which is called the **Witten deformation** of the exterior derivative. Since $\Omega^k(M)$ is linear over $C^\infty(M)$, this deformed derivative still maps $k$-forms linearly to $(k+1)$-forms. It is also clear that

$$\mathrm{d}_t^2 = (e^{-tf}\mathrm{d}e^{tf})(e^{-tf}\mathrm{d}e^{tf}) = e^{-tf}\mathrm{d}\circ\mathrm{d}\,e^{tf} = 0,$$

So $(\Omega^k(M), \mathrm{d}_t)$ defines another cochain complex with cohomology groups $H_t^k(M)$.

**Proposition 59.** *For all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, $H_{dR}^k(M)$ is isomorphic to $H_t^k(M)$, and so*

$$\dim H_{dR}^k(M) = \dim H_t^k(M)$$

*Proof.* Define the map $\phi_t : \Omega^k(M) \to \Omega^k(M)$ by $\phi_t(\omega) = e^{-tf}\omega$, which is clearly linear over $C^\infty(M)$. Note that $\mathrm{d}_t e^{-tf} = e^{-tf}\mathrm{d}$, so for any form $\omega \in \Omega^k(M)$, we have

$$\mathrm{d}_t(\phi_t(\omega)) = \mathrm{d}_t(e^{-tf}\omega) = e^{-tf}\mathrm{d}\omega = \phi_t(\mathrm{d}\omega).$$

Thus $\phi_t$ maps closed forms (under d) to closed forms (under $\mathrm{d}_t$) and exact forms (under d) to exact forms (under $\mathrm{d}_t$). Hence $\phi_t$ induces a linear map from $H_{dR}^k(M)$ to $H_t^k(M)$. But the map $\phi_{-t}$ has all the same properties and clearly inverts $\phi_t$, so the induced map is indeed an isomorphism between $H_{dR}^k(M)$ and $H_t^k(M)$ and so they must have the same dimension. □

**Proposition 60.** *The adjoint of $\mathrm{d}_t^k$ with respect to the inner product on $\Omega^k(M)$ is*

$$\mathrm{d}_t^{k*} = e^{tf}\mathrm{d}^* e^{-tf}.$$

*Proof.* Note that $\langle d_t\omega, \eta\rangle = \langle e^{-tf}\mathrm{d}e^{tf}\omega, \eta\rangle = \langle \mathrm{d}(e^{tf}\omega), e^{-tf}\eta\rangle$ by the $C^\infty(M)$-linearity, and this equals $\langle e^{tf}\omega, d^* e^{-tf}\eta\rangle = \langle \omega, e^{tf}d^*e^{-tf}\eta\rangle$ again by linearity. Hence $\mathrm{d}_t{}^* = e^{tf}\mathrm{d}^*e^{-tf}$. □

**Proposition 61.** *Let $U$ be a local chart of $M$. The exterior derivative and the codifferential are related to their deformations on $U$ as:*

$$\mathrm{d}_t = \mathrm{d} + t(\mathrm{d}f)\wedge$$
$$\mathrm{d}_t^* = \mathrm{d}^* + t(\mathrm{d}f)^\flat\lrcorner$$

*for any $t \in \mathbb{R}$ and Morse function $f$.*

*Proof.* By the Leibniz rule for the exterior derivative, we have:

$$\mathrm{d}_t\omega = e^{-tf}\mathrm{d}(e^{tf}\wedge\omega) = e^{-tf}(\mathrm{d}e^{tf})\wedge\omega + e^{-tf}e^{tf}\wedge\mathrm{d}\omega = te^{tf}(\mathrm{d}f)\wedge\omega + \mathrm{d}\omega$$

giving us the first equality. Now using **Proposition 49**, we have that:

$$\begin{aligned}
\mathrm{d}_t^*\omega &= -e^{tf}\left(\sum_{i=1}^{n}\partial_i\lrcorner\nabla_{\partial_i}\right)e^{-tf}\omega = -e^{tf}\left(\sum_{i=1}^{n}\partial_i\lrcorner\left((\partial_i e^{-tf})\omega + e^{-tf}\nabla_{\partial_i}\omega\right)\right)\\
&= -e^{tf}\left(\sum_{i=1}^{n}\partial_i\lrcorner\left((\partial_i e^{-tf})\omega\right)\right) - e^{tf}\sum_{i=1}^{n}e^{-tf}\nabla_{\partial_i}\omega\\
&= \mathrm{d}^*\omega - e^{tf}\left(\sum_{i=1}^{n}\partial_i\lrcorner\left(-te^{-tf}(\partial_i f)\omega\right)\right) = \mathrm{d}^*\omega + t\sum_{i=1}^{n}(\partial_i f)\partial_i\lrcorner\omega = \mathrm{d}^*\omega + t(\mathrm{d}f)^\flat\lrcorner\omega,
\end{aligned}$$

as required. □

We get a corresponding deformation of the Dirac operator and the Laplacian:

$$\begin{aligned}
&D_t = \mathrm{d}_t + \mathrm{d}_t^* = \mathrm{d} + t\mathrm{d}f\wedge + \mathrm{d}^* + t(\mathrm{d}f)^\flat\lrcorner = D + t\hat{c}(\mathrm{d}f);\ \text{and}\\
&\Delta_t = \mathrm{d}_t\mathrm{d}_t^* + \mathrm{d}_t^*\mathrm{d}_t = D_t^2
\end{aligned}$$

where $\hat{c}(\mathrm{d}f) = \mathrm{d}f\wedge + (\mathrm{d}f)^\flat\lrcorner$ (another Clifford operator discussed in **Appendix B**). This deformed Laplacian is in fact another self adjoint elliptic operator, so it has an analogous Hodge decomposition. This gives the following chain of isomorphisms:

$$\ker\Delta_t^k \simeq H_t^k(M) \simeq H_{dR}^k(M) \simeq \ker\Delta^k,$$

which allows us to compute the Betti numbers via studying the deformed harmonic forms. Particularly, our proof of the Morse inequalities relies on constructing a space that contains all of the harmonic forms of rank $k$, but has dimension $\mu_f(k)$.

## 4.4 Description of $\ker\Delta_t^k$

Consider the neighbourhood of a critical point $p$ of a Morse function $f$ with index $\lambda$, identifying points with their Morse coordinates $(u^1,\ldots,u^n)$ on $U$ containing $p$. Pick a metric $g$ such that $\partial_i$ form an orthonormal frame on $U$.

**Proposition 62.** *The deformed Laplacian at $u\in U$ has the local form $\Delta_t = H_t + K_t$*

*where:*

$$H_t = -\sum_{i=1}^{n}(\partial_i)^2 - nt + t^2|u|^2; \text{ and}$$
$$K_t = 2t\left(\sum_{i=1}^{\lambda}\partial_i\lrcorner \mathrm{d}u^i\wedge + \sum_{i=\lambda+1}^{n}\mathrm{d}u^i\wedge\partial_i\lrcorner\right).$$

*Proof.* From the definitions, we have that:

$$\begin{aligned}\Delta_t &= \mathrm{d}_t\mathrm{d}_t^* + \mathrm{d}_t^*\mathrm{d}_t = {D_t}^2 = (D + t\widehat{c}(\mathrm{d}f))^2 &(4.1)\\ &= D^2 + Dt\widehat{c}(\mathrm{d}f) + t\widehat{c}(\mathrm{d}f)D + t^2\widehat{c}(\mathrm{d}f)\widehat{c}(\mathrm{d}f) &(4.2)\\ &= \Delta + t(D\widehat{c}(\mathrm{d}f) + \widehat{c}(\mathrm{d}f)D) + t^2|\mathrm{d}f|^2. &(4.3)\end{aligned}$$

Since we are in a Morse chart, $|\mathrm{d}f| = |u|$. Considering the middle bracket, we have:

$$\begin{aligned}D\widehat{c}(\mathrm{d}f) + \widehat{c}(\mathrm{d}f)D &= \sum_{i=1}^{n} c(\partial_i)\nabla_{\partial_i}\widehat{c}(\mathrm{d}f) + \widehat{c}(\mathrm{d}f)c(\partial_i)\nabla_{\partial_i}\\ &= \sum_{i=1}^{n} c(\partial_i)\widehat{c}(\nabla_{\partial_i}\mathrm{d}f) + c(\partial_i)\widehat{c}(\mathrm{d}f)\nabla_{\partial_i} + \widehat{c}(\mathrm{d}f)c(\partial_i)\nabla_{\partial_i}\\ &= \sum_{i=1}^{n} c(\partial_i)\widehat{c}(\nabla_{\partial_i}\mathrm{d}f).\end{aligned}$$

But $\nabla_{\partial_i}\mathrm{d}f = (-1)^a\mathrm{d}u_i$, where $a = 1$ if $i \leq \lambda$ and $a = 0$ otherwise, thus we have:

$$D\widehat{c}(\mathrm{d}f) + \widehat{c}(\mathrm{d}f)D = -\sum_{i=1}^{\lambda} c(\partial_i)\widehat{c}(\mathrm{d}u^i) + \sum_{i=\lambda+1}^{n} c(\partial_i)\widehat{c}(\mathrm{d}u^i). \tag{4.4}$$

From the definitions, we have:

$$c(\partial_i)\widehat{c}(\mathrm{d}u^i) = (\mathrm{d}u^i\wedge -\partial_i\lrcorner)(\mathrm{d}u^i\wedge +\partial_i\lrcorner) = \mathrm{d}u^i\wedge\partial_i\lrcorner - \partial_i\lrcorner\mathrm{d}u^i\wedge.$$

Via the Leibniz rule of the interior product, we get:

$$c(\partial_i)\widehat{c}(\mathrm{d}u^i) = (\mathrm{d}u^i\wedge\partial_i\lrcorner) - (\mathrm{d}u^i(\partial_i)\wedge) + (\mathrm{d}u^i\wedge\partial_i\lrcorner) = 2\mathrm{d}u^i\wedge\partial_i\lrcorner - 1 = 1 - 2\partial_i\lrcorner\mathrm{d}u^i\wedge.$$

Inserting this into (4.4) gives:

$$\begin{aligned} D\widehat{c}(\mathrm{d}f) + \widehat{c}(\mathrm{d}f)D &= -\sum_{i=1}^{\lambda}(1 - 2\partial_i \lrcorner \mathrm{d}u^i \wedge) + \sum_{i=\lambda+1}^{n}(2\mathrm{d}u^i \wedge \partial_i \lrcorner - 1) \\ &= 2\sum_{i=1}^{\lambda} \partial_i \lrcorner \mathrm{d}u^i \wedge + 2\sum_{i=\lambda+1}^{n} \mathrm{d}u^i \wedge \partial_i \lrcorner - \sum_{i=1}^{n} 1 \\ &= 2\left(\sum_{i=1}^{\lambda} \partial_i \lrcorner \mathrm{d}u^i \wedge + \sum_{i=\lambda+1}^{n} \mathrm{d}u^i \wedge \partial_i \lrcorner\right) - n. \end{aligned}$$

Thus subsituting into (4.3), we get the sought expression:

$$\Delta_t = -\sum_{i=1}^{n}(\partial_i)^2 - nt + t^2|u|^2 + 2t\left(\sum_{i=1}^{\lambda} \partial_i \lrcorner \mathrm{d}u^i \wedge + \sum_{i=\lambda+1}^{n} \mathrm{d}u^i \wedge \partial_i \lrcorner\right) = H_t + K_t.$$

□

The operator $H_t$ (acting on $\Omega(U)$) is the well known harmonic oscillator. It has a one dimensional kernel (over $\Omega^0(U)$), generated over $\mathbb{R}$ by the Gaussian function $f(u) = \exp(-t|u|^2/2)$ (c.f [4], Theorem 5.1).

**Proposition 63.** *Let $\omega \in \Omega(U)$. We have $H_t(\omega_I \mathrm{d}u^I) = (H_t\omega_I)\mathrm{d}u^I$ and if $H_t(\omega_I \mathrm{d}u^I) = 0$ then $\omega_I = af$ for some $a \in \mathbb{R}$.*

*Proof.* The first part is clear from the definition of $H_t$. The second is then an immediate corollary. □

**Proposition 64.** *Let $\omega \in \Omega(U)$. Then $K_t\omega = 2tm\omega$ for some $m \in \mathbb{N}$ and* $\ker K_t =$ $\mathrm{span}_{C^\infty(M)}(\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^\lambda) = \mathrm{span}_{C^\infty(M_p)}(\omega_0)$.

*Proof.* On $U$ we have $\omega = \omega_I \mathrm{d}u^I$ and $K_t\omega = \omega_I K_t \mathrm{d}u^I$ by the $C^\infty$-linearity of $\lrcorner$ and $\wedge$. Note that $\partial_i \lrcorner \mathrm{d}u^i \wedge \mathrm{d}u^I = 0$ if $i \in I$ and equals $\mathrm{d}u^I$ otherwise. Also see that $\partial_i \lrcorner \mathrm{d}u^I = 0$ if $i \notin I$, and if $i = i_a \in I$ then $\partial_i \lrcorner \mathrm{d}u^I = (-1)^{a+1}\mathrm{d}u^{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}u^{i_a}} \wedge \cdots \wedge \mathrm{d}u^{i_k}$. Wedging with $\mathrm{d}u^i$, we get:

$$\mathrm{d}u^i \wedge \partial_i \lrcorner \mathrm{d}u^I = (-1)^{a+1}\mathrm{d}u^{i_a} \wedge \mathrm{d}u^{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}u^{i_a}} \wedge \cdots \wedge \mathrm{d}u^{i_k} = (-1)^{a-1}(-1)^{a-1}\mathrm{d}u^I$$

Thus we have:

$$\begin{aligned} K_t(\omega_I \mathrm{d}u^I) &= 2t\left(\sum_{i=1}^{\lambda} \partial_i \lrcorner \mathrm{d}u^i \wedge + \sum_{i=\lambda+1}^{n} \mathrm{d}u^i \wedge \partial_i \lrcorner\right)\omega_I \mathrm{d}u^I \\ &= 2t\omega_I\left(\sum_{i=1}^{\lambda} \partial_i \lrcorner \mathrm{d}u^i \wedge \mathrm{d}u^I + \sum_{i=\lambda+1}^{n} \mathrm{d}u^i \wedge \partial_i \lrcorner \mathrm{d}u^I\right) \\ &= 2t\omega_I\left(a(I)\mathrm{d}u^I + b(I)\mathrm{d}u^I\right) = 2tm\omega \end{aligned}$$

where $a(I) = |\{i \in \mathbb{N} : i \leq \lambda \text{ and } i \notin I\}|$ and $b(I) = |\{i \in \mathbb{N} : \lambda < i \leq n \text{ and } i \in I\}|$, proving the first statement. Now $K_t\omega = 0$ if and only if $m = a(I) + b(I) = 0$, which is equivalent to $a(I) = b(I) = 0$. This only occurs if $I$ is some permutation of $(1, \ldots, \lambda)$. Thus the kernel of $K_t$ is generated over $C^\infty(U)$ by $\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^\lambda$. □

**Theorem 65.** $\ker \Delta_t|_U = \mathrm{span}_{\mathbb{R}}(f\omega_0)$.

*Proof.* First, note that $[K_t, H_t]\omega = K_t 2tm\omega - H_t(K_t\omega_I)\mathrm{d}u^I = 2tmK_t\omega - 2tmK_t\omega = 0$ by the above propositions and the linearity of $K_t$ and $H_t$. Since $[K_t, H_t] = 0$, $[\Delta_t, K_t] = [K_t + H_t, K_t] = 0 = [\Delta_t, H_t]$, all three are simultaneously diagonalisable from which the result follows. □

This gives us a description of harmonic forms in $\Omega(U)$:

$$\ker \Delta^k = \mathrm{span}_{\mathbb{R}}\{\exp(\frac{-t|u|^2}{2})\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^\lambda\}$$

This immediately suggests that joining these spaces together gives us some connection between the Morse numbers (which come from the number of generators of a given rank) and the Betti numbers, given by the dimension of the kernel restricted to the space of forms of a given rank. However, this adjoining process introduces some mismatch, the result of which is an inequality rather than equality.

To extend these generators from each $U$ to $M$, we want to keep the behaviour localised around $p$ by ensuring that it vanishes on $M \setminus U$, and yet remains smooth. To do this, we take a bump function $\gamma : M \to [0, 1]$ for which $\gamma(u) = 1$ on some disc of radius $r$ centred at $p$ (in the Morse co-ordinates), and smoothly decreases to 0 outside of a disc of radius $2r$ also centred at $p$ (where this disc is still entirely contained inside $U$). Define the forms:

$$\rho_{p,t} = \frac{\gamma(u)}{\sqrt{\alpha_{p,t}}}\exp(\frac{-t|u|^2}{2})\mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^\lambda \in \Omega^\lambda(M)$$

where the normalisation function $\alpha_{p,t}$ is given by

$$\alpha_{p,t} = \int_{M_p} \gamma(u) \exp(\frac{-t|u|^2}{2}) \mathrm{d}u^1 \wedge \cdots \wedge \mathrm{d}u^\lambda,$$

which ensures that $\langle \rho_{p,t}, \rho_{p,t} \rangle = 1$. These functions are in general not in the kernel of $\Delta_t$ because of the Leibniz rule. However, they are still useful for describing the kernel of $\Delta_t$ as we will show.

Define $H^0$ and $H^1$ as the $0^{th}$ and $1^{st}$ Sobolev spaces induced by the inner product on $\Omega(M)$, with norms $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively. Define the space $E_t \subseteq H^0$ as the subspace generated by the set $\{\rho_{p,t} : p \in \mathbf{Cr}_f\}$. Take $E_t^\perp$ as its orthogonal complement in $H^0$. We can then decompose the Dirac operator $D_t = \mathrm{d}_t + \mathrm{d}_t^*$ via the projections $\pi_t : H_0 \to E_t$ and $\pi_t^\perp : H^0 \to E_t^\perp$: i.e.,

$$D_{t,1} = \pi_t D_t \pi_t \qquad D_{t,2} = \pi_t D_t \pi_t^\perp$$
$$D_{t,3} = \pi_t^\perp D_t \pi_t \qquad D_{t,4} = \pi_t^\perp D_t \pi_t^\perp$$

Further, take $c > 0$ and let $E_t(c)$ be the subspace of $H^0$ defined as the direct sum of eigenspaces of $D_t$ with eigenvalues in the range $[-\sqrt{c}, \sqrt{c}]$. Let $\varpi_{t,c} : H^0 \to E_t(c)$ be the orthogonal projection onto $E_t(c)$. Using techniques outside of the scope of this dissertation, one can prove the following estimates. For a detailed proof, see [9].

**Lemma 66.** *We have that:*

1. $D_{t,1} = 0$ *for all* $t > 0$*;*
2. *there exists a* $T_1 > 0$ *such that for all* $t > T_1$*,* $s \in E_t^\perp \cap H^1$ *and* $s' \in E_t$*,*

$$\|D_{t,2}s\|_0 \leq \frac{\|s\|_0}{t} \qquad \text{and} \qquad \|D_{t,3}s'\|_0 \leq \frac{\|s'\|_0}{t};$$

3. *there exists a* $T_2 > 0$ *and a* $C_1 > 0$ *such that for all* $t > T_2$ *and* $s \in E_t^\perp \cap H^1$*,*

$$\|D_{t,4}s\|_0 \geq C_1\sqrt{t}\|s\|_0; \text{ and}$$

4. *there exists a* $T_3 > 0$ *and a* $C_2 > 0$ *such that for all* $t > T_3$ *and* $s \in E_t$*,*

$$\|\varpi_{t,C_1}s - s\|_0 \leq \frac{C_2}{t}\|s\|_0.$$

These estimates allow us to prove the following result, which we use to show that $E_t(c) \cap \overline{\Omega^k(M)}$ not only contains every harmonic form of rank $k$, but has dimension $\mu_f(k)$.

**Proposition 67.** *For any $c > 0$, there exists a $T_0 > 0$ such that for all $t > T_0$, the number of eigenvalues of $\Delta_t^k$ in the range $[0, c]$ equals $\mu_f(k)$ for all $k \in \{0, \ldots, n\}$.*

*Proof.* Take distinct $x, y \in \mathbf{Cr}_f$ and $c > 0$. Note that $\langle \rho_{x,t}, \rho_{y,t} \rangle = 0$ and so

$$\begin{aligned}\langle \varpi_{t,c}\rho_{x,t}, \varpi_{t,c}\rho_{y,t} \rangle &= \langle \varpi_{t,c}\rho_{x,t} - \rho_{x,t} + \rho_{x,t}, \varpi_{t,c}\rho_{y,t} - \rho_{y,t} + \rho_{y,t} \rangle \\ &= \langle \varpi_{t,c}\rho_{x,t} - \rho_{x,t}, \varpi_{t,c}\rho_{y,t} - \rho_{y,t} \rangle + \langle \rho_{x,t}, \varpi_{t,c}\rho_{y,t} - \rho_{y,t} \rangle \\ &\quad + \langle \varpi_{t,c}\rho_{x,t} - \rho_{x,t}, \rho_{y,t} \rangle.\end{aligned}$$

By the Cauchy-Shawrtz inequality, we have:

$$\begin{aligned}|\langle \varpi_{t,c}\rho_{x,t}, \varpi_{t,c}\rho_{y,t} \rangle| \leq{}& \|\varpi_{t,c}\rho_{x,t} - \rho_{x,t}\|_0 \|\varpi_{t,c}\rho_{y,t} - \rho_{y,t}\|_0 + \|\rho_{x,t}\|_0 \|\varpi_{t,c}\rho_{y,t} - \rho_{y,t}\|_0 \\ &+ \|\rho_{y,t}\|_0 \|\varpi_{t,c}\rho_{x,t} - \rho_{x,t}\|_0.\end{aligned}$$

Because $\|\rho_{x,t}\|_0 = \|\rho_{y,t}\|_0 = 1$ and by part 4 of **Lemma 66**, the right hand side tends to zero as $t \to 0$, and so for $t$ greater than some $T_4 > 0$, the terms $\varpi_{t,c}\rho_p$ are linearly independent for all $p \in \mathbf{Cr}_f$. Thus we have that $\dim E_t = \dim \varpi_{t,c} E_t$ and so:

$$\dim E_t \leq \dim E_t(c).$$

Suppose for a contradiction that the inequality is strict. Then there exists some non-zero $s \in E_t(c) \cap (\varpi_{t,c} E_t)^\perp$, for which:

$$\begin{aligned}\pi_t s &= \sum_{x \in \mathbf{Cr}_f} \langle s, \rho_{x,t} \rangle \rho_{x,t} \\ &= \sum_{x \in \mathbf{Cr}_f} \langle s, \rho_{x,t} \rangle \rho_{x,t} - \langle s, \rho_{x,t} \rangle \varpi_{t,c}\rho_{x,t} + \langle s, \rho_{x,t} \rangle \varpi_{t,c}\rho_{x,t} - \langle s, \varpi_{t,c}\rho_{x,t} \rangle \varpi_{t,c}\rho_{x,t} \\ &= \sum_{x \in \mathbf{Cr}_f} \langle s, \rho_{x,t} \rangle (\rho_{x,t} - \varpi_{t,c}\rho_{x,t}) + \langle s, \rho_{x,t} - \varpi_{t,c}\rho_{x,t} \rangle \varpi_{t,c}\rho_{x,t}\end{aligned}$$

Thus again using the Cauchy-Scharz inequality and noting that $\|\varpi_{t,c}\rho_{x,t}\|_0 \leq \|\rho_{x,t}\|_0 = 1$ we get:

$$\begin{aligned}\|\pi_t s\|_0 &\leq \sum_{x \in \mathbf{Cr}_f} \|s\|_0 \|\rho_{x,t}\|_0 \|\varpi_{t,c}\rho_{x,t} - \rho_{x,t}\|_0 + \|s\|_0 \|\varpi_{t,c}\rho_{x,t}\|_0 \|\varpi_{t,c}\rho_{x,t}\|_0 \\ &\leq \sum_{x \in \mathbf{Cr}_f} 2\|\varpi_{t,c}\rho_{x,t} - \rho_{x,t}\|_0 \|s\|_0.\end{aligned}$$

By part 4 of **Lemma 66** we see that with $C_2 = 2|\mathbf{Cr}_f|C_1 > 0$, then $2\sum_{x \in \mathbf{Cr}_f} \|\varpi_{t,c}\rho_{x,t} - \rho_{x,t}\|_0 \leq \frac{C_2}{t}\|\rho_{x,t}\|_0 = \frac{C_2}{t}$, and thus $\|\pi_t s\|_0 \leq \frac{C_2}{t}\|s\|_0$ for all $t > T_3$. The reverse triangle inequality then implies that

$$\|\pi_t^\perp s\|_0 = \|s - \pi_t s\|_0 \geq \|s\|_0 - \|\pi_t s\|_0 \geq (1 - \frac{C_2}{t})\|s\|_0$$

So by part 3 of **Lemma 66** we get:

$$C_1\sqrt{t}(1-\frac{C_2}{t})\|s\|_0 \le C_1\sqrt{t}\|\pi_t^{\perp}s\|_0 \le \|D_{t,4}\pi_t^{\perp}s\|_0 \le \|D_t\pi_t^{\perp}s\|_0 \le \|D_t s\|_0 + \|D_t\pi_t s\|_0$$

As $D_t\pi_t s = D_{t,1}s + D_{t,3}s = D_{t,3}s$, we have:

$$C_1\sqrt{t}C_3\|s\|_0 \le \|D_t s\|_0 + \|D_{t,3}s\|_0 \le \|D_t s\|_0 + \frac{1}{t}\|s\|_0$$

(again by **Lemma 66**) and so $\|D_t s\|_0 \ge (C_1\sqrt{t}C_3 - \frac{1}{t})\|s\|_0$. But then as $T \to \infty$ $\|D_t s\|_0 \to \infty$, contradicting the assumption that $s$ is a linear combination of eigenvectors of $D_t$ with eigenvalues in the range $[-\sqrt{c}, \sqrt{c}]$. Thus we must have $\dim E_t(c) = \dim E_t = \sum_{k=0}^{n} \mu_f(k)$, with $\{\varpi_{t,c}\rho_{x,t} : x \in \mathbf{Cr}_f\}$ as a basis for $E_t(c)$.

Now let $Q_i$ denote the projections from $H^0$ onto the completions of each $\Omega^i(M)$ in $H^0$. For each $x \in \mathbf{Cr}_f$ with index $\lambda$, we have

$$\|Q_\lambda \varpi_{t,c}\rho_{x,t} - \rho_{x,t}\|_0 = \|Q_\lambda(\varpi_{t,c}\rho_{x,t} - \rho_{x,t})\|_0 \le \|\varpi_{t,c}\rho_{x,t} - \rho_{x,t}\|_0 \le \frac{C_2}{t}.$$

Thus for all $t > T_0$, the terms $Q_\lambda\varpi_{t,c}\rho_{x,t}$ are linearly independent and so

$$\dim Q_k E_t(c) \ge \mu_f(k)$$

for all $k \in \{0, \ldots, n\}$. If the inequality were strict, then summing over $k$ would give

$$\sum_{k=0}^{n} \dim Q_k E_t(c) > \sum_{k=1}^{n} \mu_f(k) = \sum_{k=0}^{n} \dim E_t(c)$$

a clear contradiction, so $\dim Q_k E_t(c) = \mu_f(k)$. Then note that if $D_t s = as$ where $a \in [-\sqrt{c}, \sqrt{c}]$, then

$$\Delta_t Q_i s = Q_i D_t^2 s = a^2 Q_i s$$

as $\Delta_t$ preserves the grading of $\Omega(M)$. So $Q_i E_t(c)$ is the space of eigenvectors of $\Delta_t$ with eigenvalues in the range $[0, c]$, with dimension $\mu_f(k)$ for $t > T_0$. □

# Chapter 5

# The Morse Inequalities

## 5.1 Statement and Proof

We are now in a position to bring the previous results together to prove the Morse inequalities.

**Theorem 68** (The Morse Inequalities)**.** *Given a closed smooth $n$-manifold $M$ and any Morse function $f$ on $M$, the $k^{th}$ Betti number of $M$ is bounded above by the $k^{th}$ Morse number; i.e.,*

$$\mu_f(k) \geq \beta_k(M).$$

*These are known as the **weak Morse inequalities**. For each $k \in \{0, \ldots, n\}$, the **strong Morse inequalities** are:*

$$\sum_{l=0}^{k} (-1)^l \mu_f(k-l) = \mu_f(k) - \mu_f(k-1) + \cdots \pm \mu_f(0) \geq \beta_k - \beta_{k-1} + \cdots \pm \beta_0 = \sum_{l=0}^{k} (-1)^l \beta_{k-l}$$

*Proof.* Let $M$ be a compact manifold and let $f : M \to \mathbb{R}$ be a Morse function. For each $k \in \{0, \ldots, n\}$ and $c \in [0, \infty)$, define $F^c_{t,k} \subseteq \Omega^k(M)$ as the finite dimensional vector space generated by the eigenspaces of $\Delta^k_t$ with eigenvalues in $[0, c]$. By **Proposition 67** we can increase $t$ until $\dim F^c_{t,k} = \mu_f(k)$. But also, $\ker \Delta^k_t \subseteq F^c_{t,k-1}$, so

$$\beta_k(M) = \dim \ker \Delta^k_t \leq \dim F^c_{t,k-1} = \mu_f(k)$$

thus proving the weak Morse inequalities.

Then, by the Rank-Nullity theorem, we have that

$$\begin{aligned}
\mu_f(k) = \dim F^c_{t,k-1} &= \dim\ker \mathrm{d}_t^k|_{F^c_{t,k}} + \dim\operatorname{im}\mathrm{d}_t^k|_{F^c_{t,k}} \\
&= \dim \frac{\ker \mathrm{d}_t^k|_{F^c_{t,k}}}{\operatorname{im}\mathrm{d}_t^{k-1}|_{F^c_{t,k-1}}} + \dim\operatorname{im}\mathrm{d}_t^{k-1}|_{F^c_{t,k-1}} + \dim\operatorname{im}\mathrm{d}_t^k|_{F^c_{t,k}} \\
&= \beta_k(M) + \dim\operatorname{im}\mathrm{d}_t^{k-1}|_{F^c_{t,k-1}} + \dim\operatorname{im}\mathrm{d}_t^k|_{F^c_{t,k}}
\end{aligned}$$

Now taking the alternating sum up to $j \in \{0, \dots n\}$, we get:

$$\begin{aligned}
\sum_{i=0}^{j}(-1)^j\mu_f(j-i) &= \sum_{i=0}^{j}(-1)^j(\beta_{j-i}(M) + \dim\operatorname{im}\mathrm{d}_t^{j-i-1}|_{F^c_{t,j-i-1}} + \dim\operatorname{im}\mathrm{d}_t^{j-i}|_{F^c_{t,j-i}}) \\
&= \sum_{i=0}^{j}(-1)^j\beta_{j-i}(M) + \sum_{i=0}^{j}(-1)^j(\dim\operatorname{im}\mathrm{d}_t^{j-i-1}|_{F^c_{t,j-i-1}} + \dim\operatorname{im}\mathrm{d}_t^{j-i}|_{F^c_{t,j-i}}) \\
&= \sum_{i=0}^{j}(-1)^j\beta_{j-i}(M) + \dim\operatorname{im}\mathrm{d}_t^{i}|_{F^c_{t,i}} \\
&\geq \sum_{i=0}^{j}(-1)^j\beta_{j-i}(M),
\end{aligned}$$

proving the strong Morse inequalities.

□

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# Appendix A

# Homological Algebra

The following is a basic review of definitions and results for homology, but also hold for cohomology by reversing arrows. In particular, we provide a constructive proof of the Zigzag lemma which is used to form the Mayer-Vietoris sequence in **Section 3.3**.

**Definition 69.** *Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of modules over a ring $R$, and let $\{d_n\}_{n\in\mathbb{N}}$ be homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_n \circ d_{n+1} = 0$. Then we call $(C_n, d_n)_{n\in\mathbb{N}}$ a* ***chain complex****, and the $d_n$ the* ***boundary maps****. We call $\operatorname{Im} d$ the boundaries of $C_n$ and $\ker d$ the cycles of $C_n$.*

**Definition 70.** *Given a chain complex $C = (C_\bullet, d_\bullet)$, define its $k^{th}$* ***homology*** *groups as*

$$H_k(C) = \frac{\ker d_k}{\operatorname{im} d_{k+1}}$$

If a complex has only trivial homology groups, then we say that the chain complex is **exact**.

**Definition 71.** *Let $C = (C_\bullet, d_\bullet)$ and $D = (D_\bullet, \delta_\bullet)$ be chain complexes. A sequence of homomorphisms $f_n : C_n \to D_n$ is a* ***chain map*** *if each $f_n$ commutes with the boundary operators:*

$$f_{n-1} \circ d_n = \delta_{n-1} \circ f_n$$

*i.e., for which the following diagram commutes:*

$$\begin{array}{ccccccccc}
\longrightarrow & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow \\
 & \big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} & \\
\longrightarrow & D_{n+1} & \xrightarrow{\delta_{n+1}} & D_n & \xrightarrow{\delta_n} & D_{n-1} & \longrightarrow
\end{array}$$

These are important because of the following:

**Lemma 72.** *Chain maps map cycles to cycles and boundaries to boundaries.*

*Proof.* Let $c \in \ker d$. Then by the commutativity of $f$, $cf\delta = cdf = 0f = 0$, so indeed cycles are preserved. Now take $c = bd \in \mathrm{Im}d$. Then again by commutativity $cf = bdf = bf\delta \in \mathrm{Im}\delta$, so boundaries are mapped to boundaries too. □

Now consider a commutative diagram of the form:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \longrightarrow & \\
 & & \downarrow{\scriptstyle j_{n+1}} & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n-1}} & & \\
\longrightarrow & & D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} & \longrightarrow & \\
 & & \downarrow{\scriptstyle j_{n+1}} & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n-1}} & & \\
\longrightarrow & & E_{n+1} & \xrightarrow{d_{n+1}} & E_n & \xrightarrow{d_{n+1}} & E_{n-1} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

If the columns are exact, then we call this a **short exact sequence of chain complexes**. It is possible to relate the homology groups by a long exact sequence:

$$\xrightarrow{i_*} H_n(D) \xrightarrow{j_*} H_n(E) \xrightarrow{\delta} H_{n-1}(C) \xrightarrow{i_*} H_{n-1}(D) \xrightarrow{j_*}$$

To see this, note that the commutativity of the diagram means that $i$ and $j$ are chain maps, as defined above. They induce homomorphisms $i_* : H_n(C) \to H_n(D)$ and $j_* : H_n(D) \to H_n(E)$ given by, $[c] \mapsto [ci]$ and $[d] \mapsto [jd]$. All that we need is the boundary map $\delta : H_n(E) \to H_{n-1}(C)$.

Take $[e] \in H_n(E)$. Since $j$ is onto, $e = aj$ for some $a \in D$. Now consider $adj = ajd = ed = 0$, so $ad \in \ker j = \mathrm{Im}i$. But then there exists a $c \in C$ so that $ci = ad$. Moreover, $c$ is unique because $i$ is injective. Define $\delta$ by $[e] \mapsto [c]$.

**Lemma 73.** $\delta : H_n(E) \to H_{n-1}(C)$ *is a well defined homomorphism.*

*Proof.* First, note that $cdi = cid = add = 0$, so by injectivity of $i$, $c \in \ker d$.

Secondly, we need that $[c]$ as defined is an invariant of the choice of $a$. Suppose both $a$ and $a'$ have $aj = a'j = e$. Then $a - a' \in \ker j = \mathrm{Im}\, i$, so $a - a' = c'i$ for some $c' \in C$.

Thus $a' = a + c'i$, and so $a'd = ad + c'id = ci + c'di = (c + c'd)i$. But $[c] = [c + c'd]$, so indeed this does not depend on the choice of $a$.

Thirdly, note that the map does not depend on the choice of representative from the cohomology class of $e$. Take $e, e' \in [e]$, with preimages (under $j$) $a$ and $a'$ respectively. Then we must have $a' = a + a_0 d$ so $ad = a'd$, and thus we are done.

Lastly, $\delta$ is indeed a homomorphism. Let $[e_1] \mapsto [c_1]$ and $[e_2] \mapsto [c_2]$ via the choices $a_1$ and $a_2$ as above. See that $(a_1 + a_2)j = a_1 j + a_2 j = e_1 + e_2$ and that $(c_1 + c_2)i = c_1 i + c_2 i = a_1 d + a_2 d = (a_1 + a_2)d$. Then

$$([e_1] + [e_2])\delta = ([e_1 + e_2])\delta = [c_1 + c_2] = [c_1] + [c_2]$$

□

**Theorem 74** (Zigzag lemma)**.** *The following sequence of homology groups is exact.*

$$\longrightarrow H_n(D) \xrightarrow{j_*} H_n(E) \xrightarrow{\delta} H_{n-1}(C) \xrightarrow{i_*} H_{n-1}(D) \xrightarrow{j_*}$$

*Proof.* We must show that the kernel for each homomorphism is indeed the image of the prior one. We do this for each map below.

It is clear that $\mathrm{Im} i_* \subseteq \ker j_*$ because $ij = 0$ and so $j_* i_* = 0$. So let $[a] \in \ker j_*$. Then $aj = ed$ for some $e \in E$. By the sujectivity of $j$, there exists some $b \in D_{n+1}$ with $bj = e$. Then

$$(a - bd)j = aj - adj = aj - ajd = aj - ed = aj - aj = 0$$

so $a - bd \in \ker j = \mathrm{Im} i$. Thus $a - bd = ci$ for some $c \in C$, and so

$$cdi = cid = (a - bd)d = ad = 0$$

But then by the injectivity of $i$, $cd = 0$. Thus $[c]i_* = [a - bd] = [a]$, so $[a] \in \mathrm{Im} i_*$.

We now show $\mathrm{Im} j_* = \ker \delta$. First, note that if $[e] = [b]j_* \in \mathrm{Im} j_*$ then we must have $b \in \ker d$, so $bd = 0i$ (by injectivity of $i$), and so $[e]\delta = [0]$, so $\mathrm{Im} j_* \subseteq \ker \delta$. Now let $[e] \in \ker \delta$, and take $b \in D$ such that $bj = e$. Then $[c]\delta = [a] = 0$ so $a \in \ker d$, and thus $a = a'd$ for some $a' \in C$. Now

$$(b - a'i)d = bd - a'id = bd - a'di = bd - ai = bd - bd = 0$$

so $(b - a'i) \in \ker d$. Then $(b - a'i)j = bj - a'ij = bj = e$ and so $[b - a'i]j_* = [e] \in \mathrm{Im} j_*$.

Lastly, note that $\mathrm{Im}\delta \subseteq \ker i_*$ as $[a]\delta i_* = [c]i_* = [bd] = 0$. So take $[a] \in \ker i_*$, so $ai = bd$ for some $b \in D$. Then $bdj = bdj = aij = 0$, so $bj \in \ker d$. Then $\delta$ maps $[bj] \mapsto [a] \in \mathrm{Im}\delta$, finishing the proof. $\square$

# Appendix B

# Clifford Relations

The following are basic definitions and results needed from [3]. Let $(M, g)$ be a closed oriented Riemannian manifold.

**Definition 75.** *Take $v \in TM$ and $v^* \in T^*M$. Define the* ***Clifford maps*** $c(v) = (v^*\wedge) - (v\lrcorner)$, $\widehat{c}(v) = (v^*\wedge) + (v\lrcorner)$. *Also define these operators on the cotangent bundle by setting $c(v^*) = c(v)$ and $\widehat{c}(v^*) = \widehat{c}(v)$.*

**Proposition 76.** *We have:*

1. $c(v)c(w) + c(w)c(v) = -2g(v, w)$;
2. $\widehat{c}(v)\widehat{c}(w) + \widehat{c}(w)\widehat{c}(v) = 2g(v, w)$; *and*
3. $c(v)\widehat{c}(w) + \widehat{c}(w)c(v) = 0$.

**Corollary 77.** *If $\partial_i$ form an oriented orthonormal frame then $c(\partial_i)^2 = -1 = -\hat{c}(\partial_i)^2$*

**Proposition 78.** *The Levi-Civita connection $\nabla$ is compatible with the Clifford maps, i.e., $[\nabla_v, \rho(w)] = \rho(\nabla_v w)$ where $\rho = c$ or $\rho = \widehat{c}$.*