JNR monopoles and holomorphic spheres

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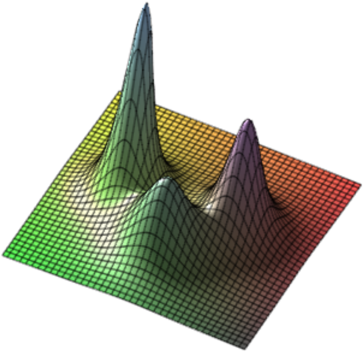
- Joint work with Paul Norbury (Melbourne)

Preliminary investigation trying to connect the Jackiw–Nohl–Rebbi (JNR) ansatze with previous work we did on hyperbolic monopoles with Michael Singer.

What is this?



- Energy density at infinity for a JNR hyperbolic monopole with poles $1, i, -1, -i$ and weights $0.9, 0.3, 0.4, 0.4$.
- I will explain what these things are and how we calculate this energy density.

Outline

- Instantons in four dimensions
- Euclidean monopoles
- Hyperbolic monopoles
- Jackiw–Nohl–Rebbi (JNR) ansatze
- Holomorphic sphere
- Energy density

Instantons in four-dimensions

Recall that if $E \to S^4$ is a bundle with connection A we say that A is an instanton if it has *self-dual* curvature that is $F_A = *F_A$.

Instantons are the minima of the Yang-Mills action

$$\mathcal{L}(A) = \int_{S^4} \|F_A\|^2 \, \mathrm{vol}_{S^4}$$

They have non-negative charge $k = c_2(E)$ and the quotient of the space of solutions by gauge transformations (autorphisms of E) is the *moduli space* which is a manifold of dimension $8k$.

The self-duality equations are invariant under conformal changes of the metric on the four-sphere.

Euclidean monopoles

Conformally $S^4 -$ point $\cong \mathbb{R}^4$. Bundles are then trivial so a connection for an $SU(2)$ bundle is an $su(2)$ valued one-form. Monopoles are *time invariant* instantons so let

$$\tilde{A}(x) = \Phi(x)dt + A(x)$$

where $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$, $\Phi\colon \mathbb{R}^3 \to su(2)$ and A is an $su(2)$ valued one-form on $\mathbb{R}^3$.

The self-duality equations become the *Bogomolny equations*

$$F_A = *d_A\Phi.$$

which minimise

$$\mathcal{L}(A, \Phi) = \int_{\mathbb{R}^3} (\|F_A\|^2 + \|d_A\Phi\|^2)d^3x$$

Φ is called the *Higgs field* and required to satisfy the boundary condition that $\|\Phi(x)\|^2 \to 1$ as $\|x\| \to \infty$.

Euclidean monopoles: existence

The boundary conditions imply that the Higgs field has a limit at infinity: $\Phi^\infty : S^2_\infty \to S^2 \subset su(2)$ so has a degree $k \geq 0$ called the (magnetic) charge of the monopole.

If $k = 1$ there is, essentially up to translations a unique monopole, the Bogomolny, Prasad, Sommerfield (BPS) monopole (1975), given by

$$\begin{aligned}
\Phi(x) &= \left(\frac{1}{r} - \frac{1}{\tanh r}\right)\frac{e}{r} \\
A(x) &= \left(\frac{1}{\sinh r} - \frac{1}{r}\right)\frac{[e, de]}{r}
\end{aligned}$$

where $r = |x|$ and $e(x) = \sum_{i=1}^{3} x^i e^i$ for an orthonormal basis e^1, e^2, e^3 of $su(2)$.

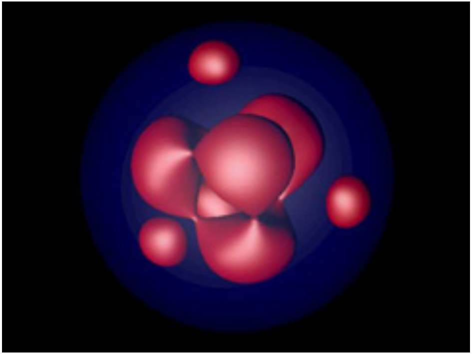
Taubes in 1980 showed the existence of monopoles of higher charge.

Energy-density isosurface

To get an idea of what a monopole looks like you plot an energy-density isosurface.

$$\|F_A\|^2 + \|d_A\Phi\|^2 = \text{constant}$$

For the BPS monopole this will be a sphere. For a higher charge monopole it might look like this $k = 7$ example:



Confession

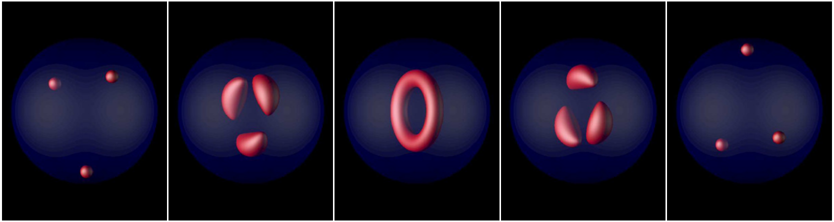
The previous picture is actually of a JNR hyperbolic monopole energy-density isosurface taken from

Stefano Bolognesi and Alex Cockburn and Paul Sutcliffe, *Hyperbolic monopoles, JNR data and spectral curves*, Nonlinearity, **28** (2015), 211–235.

But at this qualitative level hyperbolic monopoles and Euclidean monopoles look much the same. Except that the more recent the paper the nicer the pictures!

Monopole scattering

Monopoles are static things by design. But there is a hyper-kaehler metric on monopole moduli space whose geodesics can be interpreted as low-energy scattering. The moduli space is non-compact and has a region in the boundary where the energy-density isosurface will look like k widely seperated spheres. Motion along the geodesic will see the spheres come together, merge and then become separate spheres again. Here is a nice example again from Bolognesi et al of 3-monopole scattering.



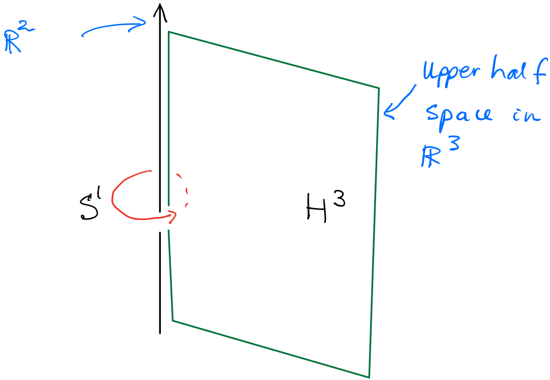
More cheating

Again I am cheating. One of the differences between Euclidean and hyperbolic monopoles is that the natural metric you can define on Euclidean monopoles is infinite when you try and define it on hyperbolic monopoles. However in the Euclidean case this picture does arise and is motion along a geodesic.

So let's think about the hyperbolic case.

Hyperbolic monopoles

Hyperbolic monopoles arise by studying a different conformal transformation $S^4 - S^2 \simeq H^3 \times S^1$ or $\mathbb{R}^4 - \mathbb{R}^2 \simeq H^3 \times S^1$.



$$g_{\mathbb{R}^4} = dx^2 + dy^2 + dt^2 + t^2 d\theta^2 = t^2 \left(\frac{dx^2 + dy^2 + dt^2}{t^2} + d\theta^2 \right)$$

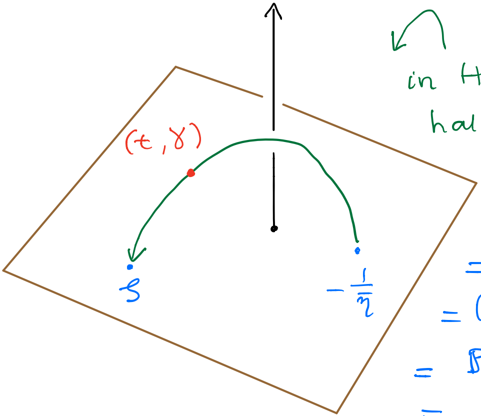
Hyperbolic monopoles

This approach was developed by Atiyah in 1984. There are two important differences to the Euclidean case:

- The length of the Higgs field Φ at infinity or m the *mass* of the monopole is a new invariant. In the Euclidean case we can trivially rescale so that $m = 1$. In the hyperbolic case that rescaling changes the curvature of H^3. If $m \in \mathbb{Z}$ the instanton on $S^4 - S^2$ extends to all of S^4.
- There is a connection at infinity which determines the monopole completely (Braam Austin 1990). In the Euclidean case the connection at infinity is always a standard form.

Note that we expect all the moduli spaces to be diffeomorphic and diffeomorphic to the Euclidean moduli space. This is known for integer mass.

Minitwistor space for hyperbolic space



oriented geodesic in H^3 in upper half-space model

All oriented geodesics = minitwistor space

$= (\eta, \zeta) : \zeta \neq -\frac{1}{\eta}$

$= \mathbb{P}_1 \times \mathbb{P}_1 - \overline{\Delta}$

$= \mathbb{M}$

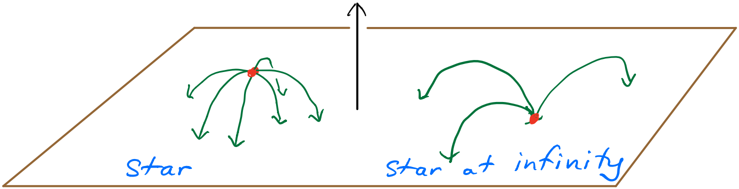
Correspondence:

$$\eta \zeta \bar{\gamma} - \eta (t^2 + |\gamma|^2) + \zeta - \gamma = 0$$

All the geodesics through (t, γ) define a curve of degree $(1, 1)$ in $\mathbb{M}$ called the *star of* (t, γ). This curve is determined by the correspondence equation

$$c_{(t,\gamma)}(\zeta, \eta) = \eta\zeta\bar{\gamma} - \eta(t^2 + |\gamma|^2) + \zeta - \gamma = 0.$$

Notice that if we let $t \to 0$ then the point (t, γ) moves out to the point γ at infinity and curve becomes $(\zeta - \gamma)(1 + \eta\bar{\gamma}) = 0$. This is the union of all the geodesics ending at γ with all the geodesics starting at γ.



Spectral curve

A monopole determines a curve S of degree (k, k) in $\mathbb{M}$ called the *spectral curve*. This means it is determined by a polynomial equation $p(\zeta, \eta) = 0$ of degree (k, k). The spectral curve also determines the monopole. It satisfies a number of constraints:

- The spectral curve is *real*. That means fixed by the involution that reverses the direction of the geodesic.
- It doesn't intersect the anti-diagonal.
- The holomorphic line bundle

$$L^{2m+k} = \mathcal{O}(2m + k, -2m - k)$$

  is trivial over S. Note that L is well-defined on $\mathbb{M}$ for non-integral values of m.

The intuition is that if we think of a monopole as a collection of k points which are very widely separated in H^3 the spectral curve is approximately the union of all the stars through those points.

Jackiw–Nohl–Rebbi ansatze

The JNR ansatze is a method of defining instantons in $\mathbb{R}^4$ (1977). It can be adapted to defining hyperbolic monopoles of mass $m = \frac{1}{2}$. You get elegant, simple, formulae for various of objects related to a monopole in particular

- the spectral curve
- the non-vanishing holomorphic section of $L^{2m+k} = L^{k+1}$
- the holomorphic sphere (not defined yet)

One reason the JNR ansatze is remarkable is that we don't have exact formulae for monopoles in terms of their connection and Higgs field or the precise polynomial defining the spectral curves except in some highly symmetric cases.

JNR data

JNR data consists of $k+1$ *poles* which are points $\gamma_0, \ldots, \gamma_k$ in S^2 and $k+1$ *charges* which are positive real numbers $\lambda_0, \ldots, \lambda_k$. The charges are only relevant up to an overall scale.

Not all monopoles arise from JNR data as it contains $2(k+1)+(k+1)-1 = 3k+2$ real parameters whereas the monopole moduli space is known to be $4k-1$ dimensional.

JNR spectral curve

The equation of the spectral curve is (Bologonesi et al)

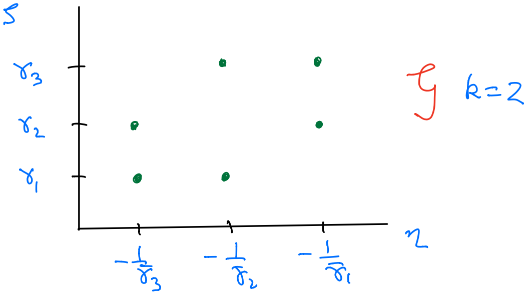
$$p(\eta,\zeta) = \sum_{i=0}^{k} \lambda_i^2 \prod_{\substack{j=0 \\ j\neq i}}^{k} (\zeta - \gamma_j)(1 + \eta\bar{\gamma}_j)$$

Notice that this has degree (k,k). I leave as an exercise checking that it is real and avoids the anti-diagonal.

Which spectral curves arise ? We can give the following partial answer.

Notice first that $p(\frac{-1}{\bar{\gamma}_i}, \gamma_j) = 0$ for any $i \neq j$.

So the JNR spectral curve vanishes on a set like



Grids

Definition

Let us define a *grid* in $\mathbb{M}$ to be a subset of the form

$$\mathcal{G} = \left\{ \left(\frac{-1}{\bar{\gamma}_i}, \gamma_j \right) \mid 0 \le i \ne j \le k \right\}$$

for $\gamma_0, \ldots, \gamma_k$ points in $\mathbb{P}_1$.

We say that a curve S in $\mathbb{M}$ *admits a grid* if there is a grid $\mathcal{G}$ with $\mathcal{G} \subset S$.

Then we have:

Proposition (M–Norbury)

Let S be a real curve of degree (k, k) not intersecting the anti-diagonal and admitting a grid $\mathcal{G}$ defined by $\gamma_0, \ldots, \gamma_k$. Then there exist unique positive real numbers $\lambda_0, \ldots, \lambda_k$ such that S is a JNR spectral curve for the poles $\gamma_0, \ldots, \gamma_k$ and weights $\lambda_0, \ldots, \lambda_k$.

Besides this result we don't have any intuition for what is special about the $3k + 2$ dimensional JNR monopole submanifold inside the full $4k - 1$ dimensional monopole moduli space.

The holomorphic section for JNR monopole

In the case that $m = \frac{1}{2}$ we want a non-vanishing section of L^{k+1}. This is given by

$$s(\eta, \zeta) = \frac{\prod_{i=0}^{k}(\zeta - \gamma_i)}{\prod_{i=0}^{k}(1 + \eta\bar{\gamma}_i)}$$

which is a meromorphic section of L^{k+1} on $\mathbb{M}$ but actually holomorphic on the spectral curve.

This is relevant to definining the *rational map* of the monopole which also can be defined using JNR data and is given in Bolognesi et al. These isn't time to discuss this here.

The JNR spectral curve formula has some nice asymptotic behaviour.

$$p(\eta, \zeta) = \sum_{i=0}^{N} \lambda_i^2 \prod_{\substack{j=0 \\ j \neq i}}^{N} (\zeta - \gamma_j)(1 + \eta \bar{\gamma}_j)$$

Rescale so that $\lambda_d = 1$ and let all the others approach zero. Then this curve approaches

$$\prod_{\substack{j=0 \\ j \neq d}}^{N} (\zeta - \gamma_j)(1 + \eta \bar{\gamma}_j).$$

which corresponds to stars at infinity through the points $\gamma_0, \ldots, \gamma_{d-1}, \gamma_{d+1}, \ldots, \gamma_k$.

So as we take this limit our monopole seems to be approximating a monopole located at points which are near to the $\gamma_0, \ldots, \gamma_{d-1}, \gamma_{d+1}, \ldots, \gamma_k$ at infinity.

Holomorphic sphere

In M–Norbury–Singer 2003 we introduced the following construction. Let

$$p(\zeta,\eta) = (-1)^k(\zeta^k, -\zeta^{k-1}, \cdots, (-1)^k)\ \Psi\ (1,\eta,\ldots,\eta^k)^t.$$

This defines a positive definite matrix Ψ.

So we can write $\Psi = Q^*Q$ for some invertible Q unique up to multiplying on the left by elements of $U(k+1)$. Define

$$q(z) = Q(1,z,\ldots,z^k)^t.$$

Then

$$q\colon \mathbb{P}^1 \to \mathbb{P}^k$$

is an embedding, unique up to the action of $U(k+1)$ called the *holomorphic sphere*. The holomorphic sphere is an invariant of the monopole.

The holomorphic sphere of a JNR monopole

In the case of a JNR monopole a straightforward calculation shows

Proposition (M-Norbury)

The holomorphic sphere of a JNR monopole with poles $\gamma_0, \dots, \gamma_k$ and weights $\lambda_0, \dots, \lambda_k$ is given by

$$q(z) = \left[\frac{\lambda_0}{z - \gamma_0}, \frac{\lambda_1}{z - \gamma_1}, \cdots, \frac{\lambda_k}{z - \gamma_k} \right].$$

Curvature of the connection at infinity

Recall that in the hyperbolic case there is a connection at infinity which determines the monopole.

In M–Norbury–Singer we showed that the curvature of the connection at infinity F_{A^∞} is given by the pull-back of the Kaehler form on $\mathbb{P}_k$ under the holomorphic sphere $q\colon \mathbb{P}_1 \to \mathbb{P}_k$.

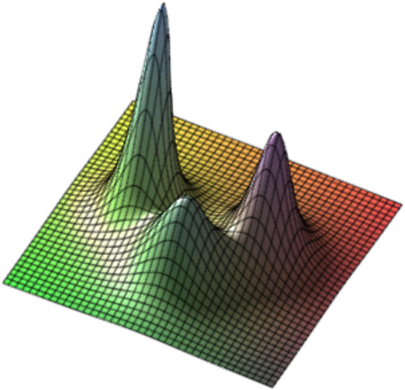
It is of interest to consider the energy density which is the function f such that

$$F_{A^\infty} = f \,\mathrm{vol}_{S^2}$$

and also the energy-density of the holomorphic sphere $q\colon \mathbb{P}_1 \to \mathbb{P}_k$. This is a straight-forward calculation and we obtain the example we began with.

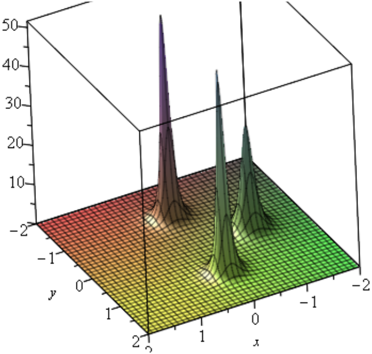
Energy-density

If we choose JNR poles $1, i, -1, -i$ and weights $0.9, 0.3, 0.4, 0.4$ then the charge density at infinity is give by



Experiments

Set $\lambda_0 = 1$ and let the others become small. Then the monopole with poles $1, i, -1, -i$ and weights $1, 0.1, 0.1, 0.1$ should be approaching $i, -1, -i$ at infinity. The energy density in fact peaks at $i, -1, -i$.



We can prove this behaviour in general.

Information metric

The holomorphic sphere and the relationship with the energy density first appeared in Braam-Austin (1990) in the case of integer mass hyperbolic monopoles. The approach in M-Norbury-Singer was a different method applicable to monopoles of any mass.

In Braam-Austin they consider the metric on monopole moduli space obtained by imposing an L^2 metric on the space of connections at infinity.

There is another natural metric on any space of probability densities called the Fisher-information metric. Future work will include trying to understand the metric on the monopole moduli space, or at least on the JNR part of it, obtained by imposing the information metric on the charge densities.

The End

- M.F. Atiyah.
  Magnetic monopoles in hyperbolic space.
  Proc. Bombay colloq. on vector bundles on algebraic varieties, 1-34 (1987).
- P. J. Braam and D. M. Austin, *Boundary values of hyperbolic monopoles*, Nonlinearity **3** (1990), no. 3, 809–823.
- Stefano Bolognesi and Alex Cockburn and Paul Sutcliffe, *Hyperbolic monopoles, JNR data and spectral curves*, Nonlinearity, **28**(1), 211–235.
- Alexander Hugh Cockburn, *Aspects of Vortices and Hyperbolic Monopoles*, PhD thesis, Durham University, 2015. http://etheses.dur.ac.uk/1107
- M. K. Murray.
  The information metric on rational maps.
  Experiment. Math., 2(4):271–279, 1993.
- M. K. Murray, P. Norbury, and M. A. Singer.
  Hyperbolic monopoles and holomorphic spheres.
  Ann. Global Anal. Geom., 23(2):101–128, 2003.