Conformal Invariant and Fixed Point Theorem

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Main Results

We use tools of noncommutative geometry to obtain:

- Local index formula of Atiyah-Singer in the setting of the action of a group of conformal-diffeomorphisms.
- Construct a class of conformal invariants out of equivariant characteristic classes.

Main References

- Ponge, R.; Wang, H.: *Noncommutative geometry and conformal geometry. I. Local index formula and conformal invariants*. J. Noncommut. Geom. (to appear).
- Ponge, R.; Hang, W.: *Noncommutative geometry and conformal geometry. II. Connes-Chern character and the local equivariant index theorem.* J. Noncommut. Geom. 10, no. 1, (2016) 303–374.

Spectral Triples

Definition (Connes)

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. $[D, a]$ is bounded for all $a \in \mathcal{A}$.

Example (Dirac Spectral Triple)

- (M^n, g) compact Riemannian spin manifold (n even) with spinor bundle $\$ = \$^+ \oplus \$^-$.
- $D\!\!\!/_g : C^\infty(M, \$) \to C^\infty(M, \$)$ is the Dirac operator of (M, g).

Then $\left(C^\infty(M), L^2_g(M, \$), D\!\!\!/_g\right)$ is a spectral triple

Overview of Noncommutative Geometry

Classical	**NCG**
Riemannian Manifold (M, g)	Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$
Vector Bundle E over M	Projective Module $\mathcal{E}$ over $\mathcal{A}$ $\mathcal{E} = e\mathcal{A}^q, \ \ e \in M_q(\mathcal{A}), \ e^2 = e$
Connection ∇^E on E	Connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$
$\mathrm{ind} \not{D}_{\nabla^E} := \mathrm{ind} \not{D}^+_{\nabla^E}$	$\mathrm{ind}\, D_{\nabla^{\mathcal{E}}} := \mathrm{ind}\, D^+_{\nabla^{\mathcal{E}}}$
de Rham Homology/Cohomology	Cyclic Cohomology/Homology
Atiyah-Singer Index Formula $\mathrm{ind} \not{D}_{\nabla^E} = \int_M \hat{A}(R^M) \wedge \mathrm{Ch}(E)$	Connes-Chern Character $\mathrm{Ch}(D)$ $\mathrm{ind}\, D_{\nabla^{\mathcal{E}}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$
Local Index Theorem	CM cocycle

Conformal Changes of Metrics

Setup

- $\left(C^\infty(M), L^2_g(M,\$), D\!\!\!/_g\right)$ is a Dirac spectral triple.
- Conformal change of metric: $\hat{g} = k^{-2}g$, $k \in C^\infty(M)$, $k > 0$.

Observation

Define $U : L^2_g(M,\$) \to L^2_{\hat{g}}(M,\$)$ *by*

$$Uf = k^{\frac{n}{2}} f \quad \forall f \in L^2_g(M,\$).$$

Then U *is a unitary operator and intertwines the spectral triples*

$$\left(C^\infty(M), L^2_{\hat{g}}(M,\$), D\!\!\!/_{\hat{g}}\right) \quad \text{and} \quad \left(C^\infty(M), L^2_g(M,\$), \sqrt{k}D\!\!\!/_g\sqrt{k}\right).$$

In particular,

$$UD\!\!\!/_{\hat{g}}U^* = \sqrt{k}D\!\!\!/_g\sqrt{k}.$$

Twisted Spectral Triples

Definition (Connes-Moscovici)

A twisted *spectral triple* $(\mathcal{A}, \mathcal{H}, D)(\mathcal{A}, \mathcal{H}, D)_\sigma$ consists of

1. A $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$.
2. An involutive algebra $\mathcal{A}$ represented in $\mathcal{H}$ together with an automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that $\sigma(a)^* = \sigma^{-1}(a^*)$ for all $a \in \mathcal{A}$.
3. A selfadjoint unbounded operator D on $\mathcal{H}$ such that
    1. D maps $\mathcal{H}^\pm$ to $\mathcal{H}^\mp$.
    2. $(D \pm i)^{-1}$ is compact.
    3. [D,a]$[D, a]_\sigma := Da - \sigma(a)D$ is bounded for all $a \in \mathcal{A}$.

Conformal Deformations of Spectral Triples

Example (Connes-Moscovici)

- An *ordinary* spectral triple $(\mathcal{A}, \mathcal{H}, D)$.
- A positive element $k \in \mathcal{A}$ with inner automorphism $\sigma(a) = k^2 a k^{-2}$, $a \in \mathcal{A}$.

Then $(\mathcal{A}, \mathcal{H}, kDk)_\sigma$ is a *twisted* spectral triple.

Example

1. Conformal Dirac spectral triple (Connes-Moscovici).
2. Twisted spectral triples over NC tori associated with "conformal weights" (Connes-Tretkoff).
3. Twisted spectral triples associated to some quantum statistical systems (e.g., Connes-Bost systems, supersymmetric Riemann gas) (Greenfield-Marcolli-Teh '13).

Twisted Noncommutative Geometry

Untwisted	**Twisted (C+M, P+W)**
Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$	Twisted Spectral Triple $(\mathcal{A}, \mathcal{H}, D)_\sigma$
Projective Module $\mathcal{E}$ over $\mathcal{A}$	Projective Module $\mathcal{E}$ over $\mathcal{A}$ w/ σ-translate $\mathcal{E}^\sigma$
Connection on $\mathcal{E}$ $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes \mathcal{L}(\mathcal{H})$	σ-Connection on $\mathcal{E}$ $\nabla^{\mathcal{E}} : \mathcal{E} \to \mathcal{E}^\sigma \otimes \mathcal{L}(\mathcal{H})$
$D_{\nabla^\mathcal{E}} : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H} \otimes \mathcal{E}$	$D_{\nabla^\mathcal{E}} : \mathcal{H} \otimes \mathcal{E} \to \mathcal{H} \otimes \mathcal{E}^\sigma$
ind $D_{\nabla^\mathcal{E}}$	ind $D_{\nabla^\mathcal{E}}$
Connes-Chern Character $\mathrm{Ch}(D)$ ind $D_{\nabla^\mathcal{E}} = \langle \mathrm{Ch}(D), \mathrm{Ch}(\mathcal{E}) \rangle$	Connes-Chern Character $\mathrm{Ch}(D)_\sigma$ ind $D_{\nabla^\mathcal{E}} = \langle \mathrm{Ch}(D)_\sigma, \mathrm{Ch}(\mathcal{E}) \rangle$
CM Cocycle	????

Conformal Dirac Spectral Triple

Setup

1. M^n is a compact spin oriented manifold (n even).
2. $\mathcal{C}$ is a conformal structure on M.
3. G is a group of conformal diffeomorphisms preserving $\mathcal{C}$. Thus, given any metric $g \in \mathcal{C}$ and $\phi \in G$,

$$\phi_* g = k_\phi^{-2} g \text{ with } k_\phi \in C^\infty(M),\ k_\phi > 0.$$

4. $C^\infty(M) \rtimes G$ is the crossed-product algebra, i.e.,

$$C^\infty(M) \rtimes G = \left\{ \sum f_\phi u_\phi;\ f_\phi \in C^\infty(M) \right\},$$
$$u_\phi^* = u_\phi^{-1} = u_{\phi^{-1}}, \qquad u_\phi f = (f \circ \phi^{-1}) u_\phi.$$

Conformal Dirac Spectral Triple

Lemma (Connes-Moscovici)

For $\phi \in G$ *define* $U_\phi : L^2_g(M, \not{S}) \to L^2_g(M, \not{S})$ *by*

$$U_\phi \xi = k_\phi^{\frac{n}{2}} \phi_* \xi \quad \forall \xi \in L^2_g(M, \not{S}).$$

Then U_ϕ *is a unitary operator, and*

$$U_\phi \not{D}_g U_\phi^* = \sqrt{k_\phi} \not{D}_g \sqrt{k_\phi}.$$

Theorem (Connes-Moscovici)

The datum of any metric $g \in \mathcal{C}$ *defines a twisted spectral triple* $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g \right)_{\sigma_g}$ *given by*

1. *The Dirac operator* $\not{D}_g$ *associated to* g.
2. *The representation* $f u_\phi \to f U_\phi$ *of* $C^\infty(M) \rtimes G$ *in* $L^2_g(M, \not{S})$.
3. *The automorphism* $\sigma_g(f u_\phi) := k_\phi f u_\phi$.

Conformal Connes-Chern Character

Theorem (RP+HW)

1. *The Connes-Chern character* $\mathrm{Ch}(\not{D}_g)_{\sigma_g} \in \mathrm{HP}^0(C^\infty(M) \rtimes G)$ *is an invariant of the conformal class* $\mathcal{C}$.
2. *For any even cyclic homology class* $\eta \in \mathrm{HP}_0(C^\infty(M) \rtimes G)$, *the pairing,*

$$\langle \mathrm{Ch}(\not{D}_g)_{\sigma_g}, \eta \rangle,$$

is a scalar conformal invariant.

Computation of $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$

Theorem (Ferrand, Obata)

If the conformal structure $\mathcal{C}$ is non-flat, then G is a compact Lie group, and so $\mathcal{C}$ contains a G-invariant metric.

Fact

If $g \in \mathcal{C}$ be G-invariant, then $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g\right)_{\sigma_g}$ is an ordinary spectral triple (equivariant Dirac spectral triple, $\sigma_g = 1$).

Consequence

When $\mathcal{C}$ is non-flat, we are reduced to the computation of the Connes-Chern character of $\left(C^\infty(M) \rtimes G, L^2_g(M, \not{S}), \not{D}_g\right)$ where G is a group of isometries.

Local Index Formula in Conformal Geometry

Setup

- $\mathcal{C}$ is a nonflat conformal structure on M.
- g is a G-invariant metric in $\mathcal{C}$.

Notation

Let $\phi \in G$. Then

- M^ϕ is the fixed-point set of ϕ; this is a disconnected sum of submanifolds.
  $M^\phi = \bigsqcup M_a^\phi, \quad \dim M_a^\phi = a.$
- $\mathcal{N}^\phi = (TM^\phi)^\perp$ is the normal bundle (vector bundle over M^ϕ).

Local Index Formula in Conformal Geometry

Theorem (RP+HW)

Let g be any G-invariant metric in $\mathcal{C}$,

1. *The Connes-Chern character $\mathrm{Ch}(\not{D}_g)_{\sigma_g}$ is represented by the CM cocycle $\varphi^{\mathrm{CM}} = (\varphi^{\mathrm{CM}}_{2q})$.*
2. *We have*

$$\varphi^{\mathrm{CM}}_{2q}(f^0 u_{\phi_0}, \cdots, f^{2q} u_{\phi_{2q}}) =$$
$$\frac{(-i)^{\frac{n}{2}}}{(2q)!} \sum_a (2\pi)^{-\frac{a}{2}} \int_{M^\phi_a} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi \left(R^{\mathcal{N}^\phi}\right) \wedge f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^{2q},$$

where $\phi := \phi_0 \circ \cdots \circ \phi_{2q}$, and $\tilde{f}^j := f^j \circ \phi_0^{-1} \circ \cdots \circ \phi_{j-1}^{-1}$, and

$$\nu_\phi \left(R^{\mathcal{N}^\phi}\right) := \det{}^{-\frac{1}{2}} \left[1 - \phi'_{|\mathcal{N}^\phi} e^{-R^{\mathcal{N}^\phi}}\right].$$

Local Index Formula in Conformal Geometry

Remark

The n-th degree component is given by

$$\varphi_n(f^0 U_{\phi_0}, \cdots, f^n U_{\phi_n}) = \begin{cases} \int_M f^0 d\tilde{f}^1 \wedge \cdots \wedge d\tilde{f}^n & \text{if } \phi_0 \circ \cdots \circ \phi_n = 1, \\ 0 & \text{if } \phi_0 \circ \cdots \circ \phi_n \neq 1. \end{cases}$$

This represents Connes' transverse fundamental class of M/G.

Remark

- The computation of the CM cocycle is obtained as a consequence of a new heat kernel proof of the equivariant index theorem of Atiyah-Segal-Singer.
- It combines Getzler's rescaling with the Greiner-Hadamard approach to the heat kernel asymptotics.

Mixed Complexes

Definition (Kassel)

A mixed complex is given by

1. Spaces of chains C_m, $m \in \mathbb{N}_0$.
2. Differentials $b : C_\bullet \to C_{\bullet-1}$ and $B : C_\bullet \to C_{\bullet+1}$.

Definition (Kassel)

The mixed (or cyclic) homology of a mixed complex $(C_\bullet, b, B)$ is the homology of the complex,

$$(CC_\bullet, b + B), \quad CC_m := \bigoplus_{p+q=m} C_{p-q} = C_m \oplus C_{m-1} \oplus \cdots .$$

Mixed Complexes

Example (Connes, Tsygan)

The cyclic homology of a unital algebra $\mathcal{A}$ is the mixed homology of the mixed complex,

$$(C_\bullet(\mathcal{A}), b, B), \quad C_m(\mathcal{A}) = \mathcal{A}^{\otimes(m+1)}.$$

Example

Any cyclic object C (i.e., simplicial object with a cyclic structure) in any Abelian category gives rise to a mixed complex,

$$(C_\bullet, b, B), \quad B = (1-t)sN, \quad N_m = 1 + t + \cdots + t^m.$$

Group Homology

Fact

Any (discrete) group Γ admits the free resolution,

$$\cdots \xrightarrow{\partial} \mathbb{C}[\Gamma]^{\otimes 3} \xrightarrow{\partial} \mathbb{C}[\Gamma]^{\otimes 2} \xrightarrow{\partial} \mathbb{C}[\Gamma] \longrightarrow \mathbb{C} \rightarrow 0,$$

$$\partial(\psi_0, \ldots, \psi_m) = \sum_{0 \leq j \leq m} (-1)^j (\psi_0, \ldots, \hat{\psi}_j, \ldots, \psi_m), \quad \psi_j \in \Gamma.$$

Definition

The group homology of Γ with coefficients in a Γ-module V is the homology of the complex,

$$(\mathbb{C}[\Gamma]^{\otimes(\bullet+1)} \otimes_\Gamma V, \partial \otimes 1).$$

Remark

For $V = \mathbb{C}$ we get the group homology $H_\bullet(\Gamma) \simeq H_\bullet(B\Gamma)$.

Equivariant Cohomology

Definition (Bott, Borel)

Given an action of Γ on a manifold X, the equivariant cohomology $H^\bullet_\Gamma(X)$ is the homology of the double complex,

$$\left(\mathrm{Hom}_\Gamma\left(\mathbb{C}[\Gamma]^{\otimes(p+1)}, \Omega^q(X)\right), \partial, d\right).$$

Remark (Borel, Dupont)

We have $H^\bullet_\Gamma(X) \simeq H^\bullet(E\Gamma \times_\Gamma X)$.

Theorem (Bott)

Given a Γ-equivariant bundle E over X and a connection ∇ on E, there is a well defined equivariant Chern character,

$$\mathrm{Ch}_\Gamma(E, \nabla) \in H^\bullet_\Gamma(X).$$

Mixed Equivariant Homology

Definition

The mixed equivariant homology $H^{\Gamma}_{\bullet}(X)^{\natural}$ is the mixed homology of the mixed double complex,

$$\left(\mathbb{C}[\Gamma]^{\otimes(\bullet+1)} \otimes \Omega^{\bullet}(X), \partial, d\right).$$

That is, the mixed homology of the total mixed complex,

$$\left(\bigoplus_{p+q=\bullet} \mathbb{C}[\Gamma]^{\otimes(p+1)} \otimes \Omega^{q}(X), \partial, d\right).$$

Proposition

There is a natural bilinear map,

$$H^{\bullet}_{\Gamma}(X) \times H_{\bullet}(\Gamma) \longrightarrow H^{\Gamma}_{\bullet}(X)^{\natural}.$$

Construction of Conformal Invariants

Theorem (RP+HW)

Given $\phi \in G$, let $G_\phi = \{\psi \in G; \psi \circ \phi = \phi \circ \psi\}/\langle \phi \rangle$ be its normalizer *(where $\langle \phi \rangle$ is the subgroup generated by ϕ). Then*

1. *There is an explicit embedding,*

$$H_\bullet^{G_\phi}(M^\phi)^\natural \hookrightarrow HC_\bullet(\mathcal{A}_G), \quad \mathcal{A}_G := C^\infty(M) \rtimes G.$$

2. *We have a bilinear map,*

$$H^\bullet_{G_\phi}(M^\phi) \times H_\bullet(G_\phi) \ni (\omega, \gamma) \longrightarrow \eta(\omega, \gamma) \in HC_\bullet(\mathcal{A}_G).$$

3. *For any $(\omega, \gamma) \in H^\bullet_{G_\phi}(M^\phi) \times H_\bullet(G_\phi)$, the pairing*

$$I_g^\phi(\omega, \gamma) := \langle \mathrm{Ch}(\not{D}_g)_{\sigma_g}, \eta(\omega, \gamma) \rangle$$

is an invariant of the conformal class $\mathcal{C}$.

Conformal Invariants

Theorem (RP+HW)

Let E be G-equivariant vector bundle over M and ∇^E a G-invariant connection on E. Then

1. *For G-invariant metric $g \in \mathcal{C}$,*

$$I_g^\phi\left(\mathrm{Ch}_{\Gamma_\phi}(E_{|M^\phi}, \nabla^E), 1\right) = \int_{M^\phi} \hat{A}(R^{TM^\phi}) \wedge \nu_\phi\left(R^{\mathcal{N}^\phi}\right) \wedge \mathrm{Ch}(F^E),$$
$$= \mathrm{ind}\, D\!\!\!/_{\nabla^E}(\phi),$$

   where $\mathrm{ind}\, D\!\!\!/_{\nabla^E}(\phi)$ is the equivariant index of $D\!\!\!/_{\nabla^E}$ at ϕ.

2. *When ϕ has isolated fixed-points we recover the Lefschetz number of ϕ w.r.t. the spin complex (with coefficients in E).*

Remarks & Consequences

Corollary (Branson-Ørsted; Conformal Indices)

Given $\phi \in G$, let $g \in \mathcal{C}$ be such that $\phi_* g = g$. Then the value of

$$\mathrm{Pf}_{t \to 0+} \operatorname{Tr}\left[\gamma e^{-t D\!\!\!/_g^2} U_\phi\right] = \varphi_0^{\mathrm{CM}}(1) = \langle \mathrm{Ch}(D\!\!\!/_g)_{\sigma_g}, 1 \rangle$$

does not depend on the choice of g.

Remark

The results of this talk provide us with a cohomological interpretation of Branson-Ørsted's result. In particular, the conformal indices are conformal invariants and Fredholm indices in the usual sense.