# C-projective Equivalence in Kähler Geometry 

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- Recently, there has been new interest in c-projective geometry:
- Hamiltonian 2-forms (Apostolov-Calderbank-Gauduchon-Tonnesen-Friedman)
- C-projective transformations of Kähler manifolds (Matveev, Kiosak, Rosemann,...)
- Integrable systems (Kiyohara, Topalov, Matveev,...)
- Parabolic geometries


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- Parabolic geometries
- My talk is based on joint papers arXiv:1512.04516, 1705.11138 with Calderbank-Eastwood-Matveev, and with Matveev.

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Equivalently, if $\exists$ a 1-form $\Upsilon \in \Omega^{1}(M)$ such that

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Projective manifolds can be viewed as geometric structures infinitesimally modelled on

$$
\mathbb{R P}^{n} \cong \operatorname{PSL}(n+1, \mathbb{R}) / P
$$

via Cartan connections.

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A Cartan geometry of type $(G, P)$ on a manifold $M$ is given by:

- a principal $G$-bundle $\tilde{\mathcal{G}} \rightarrow M$ with a principal connection $\tilde{\omega} \in \Omega^{1}(\tilde{\mathcal{G}}, \mathfrak{g})$
- a reduction of structure group $i: \mathcal{G} \rightarrow \tilde{\mathcal{G}}$ to $P$ that satisfy that $\omega=i^{*} \tilde{\omega} \in \Omega^{1}(\mathcal{G}, \mathfrak{g})$ induces an isomorphism

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Homogenous model: $G \rightarrow G / P$ (with $\left.\tilde{G}:=G \times_{P} G \cong G / P \times G\right)$ equipped with the Maurer-Cartan form $\omega=\omega_{M C} \in \Omega^{1}(G, \mathfrak{g})$

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If $G$ is semisimple and $P$ a parabolic subgroup, then a Cartan geometry of type $(G, P)$ is called a parabolic geometry.

Curvature: $\kappa \in \Omega_{\text {hor }}^{2}(\mathcal{G}, \mathfrak{g})^{P} \cong \Omega^{2}\left(M, \mathcal{G} \times{ }_{P} \mathfrak{g}\right)$.
$\kappa \equiv 0$ if and only if the Cartan geometry is locally equivalent to its homogeneous model.

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For any representation $\mathbb{V}$ of $G$ the Cartan connection induces a linear connection (tractor connection) $\nabla^{\mathcal{V}}$ on

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Examples:

- Riemannian manifolds $\leftrightarrow$ torsion-free Cartan geometries of type $\operatorname{Euc}(n) / O(n) \cong \mathbb{R}^{n}$
- Projective manifolds $\leftrightarrow$ normal Cartan geometries of type $\operatorname{PSL}(n+1, \mathbb{R}) / P \cong \mathbb{R} \mathbb{P}^{n}$
- Conformal manifolds $\leftrightarrow$ normal Cartan geometries of type $\mathrm{SO}(n+1,1) / P \cong S^{n}$

Metrisability of projective structures (Liouville)
Given a projective structure ( $M^{n},[\nabla]$ ), does $[\nabla]$ contain the Levi-Civita connection of a (pseudo-)Riemannian metric? If so, how many compatible metrics are there?

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## Question

Given a (pseudo-)Riemannian manifold, what are the geometric and topological consequences of degree of mobility at least 2?
$\leadsto$ commuting integrals for the geodesic flow
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$\leadsto$ special metric projective invariant connections on tractor bundles (Kiosak-Matveev, Gover-Matveev, CEMN)
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## Remark

- Parabolic geometries, such as projective structures, come equipped with sequences of invariant differential operators, called BGG-sequences-a sequence for each tractor bundle (Čap-Slovak-Soucek, Calderbank-Diemer),
- The first operators of these sequences gives rise to linear overdetermined systems of PDE, e.g. the metrisability equation of projective structures and the equations for any kind of Killing tensors on Riemannian manifolds.

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- A curve $c:(a, b) \rightarrow M$ is called $J$-planar with respect to a complex affine connection $\nabla$ (i.e. $\nabla J=0$ ), if $\nabla_{\dot{c}} \dot{c}$ lies in the span of $\dot{c}$ and $J \dot{c}$.


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- Two complex connections are called c-projectively equivalent, if they have the same $J$-planar curves.


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- Two complex connections are called c-projectively equivalent, if they have the same $J$-planar curves.

Example: Consider complex projective space $\mathbb{C P}^{n}$ equipped with the c-projective structure induced by (the Levi-Civita connection of) the Fubini-Study metric $g_{\text {FS }}$. Then the J-planar curves are the regular (smooth) curves that lie within complex lines.

Proposition (Otsuki/Tashiro, Mikeš/Sinyukov)
Two torsion-free affine complex connections $\nabla$ and $\hat{\nabla}$ on ( $M, J$ ) are $c$-projectively equivalent $\Longleftrightarrow$ if $\exists$ a 1 -form $\Upsilon$ on $M$ such that

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\hat{\nabla}_{a} X^{c}=\nabla_{a} X^{c}+v_{a b}^{c} X^{b},
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where

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v_{a b}{ }^{c}:=\frac{1}{2}\left(\Upsilon_{a} \delta_{b}{ }^{c}+\delta_{a}{ }^{c} \Upsilon_{b}-J_{a}{ }^{d} \Upsilon_{d} J_{b}{ }^{c}-J_{b}{ }^{d} \Upsilon_{d} J_{a}{ }^{c}\right) .
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A c-projective structure on $M^{2 n}(2 n \geq 4)$ consists of a complex structure $J$ and a c-projective equivalence class of complex torsion-free connections [ $\nabla$ ].

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C-projective densities: We set $\mathcal{E}(1):=\left(\wedge^{2 n} T M\right)^{\frac{1}{n+1}}$ and $\mathcal{E}(-1):=\mathcal{E}(1)^{*}$. It follows that on $\mathcal{E}(1)$ one has

$$
\hat{\nabla}_{\mathrm{a}} \Sigma=\nabla_{\mathrm{a}} \Sigma+\Upsilon_{\mathrm{a}} \Sigma
$$

$\leadsto$ special connections in c-projective class

Set $G:=\operatorname{PSL}(n+1, \mathbb{C})$ and let $P$ be the stabiliser in $G$ of a line in $\mathbb{C}^{n+1}$.

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Theorem (Yoshimatsu, 1978)
There is an equivalence of categories between c-projective manifolds $\left(M^{2 n}, J,[\nabla]\right)(2 n \geq 4)$ and (real) normal torsion-free parabolic geometries of type $(G, P)$. The flat homogeneous model is

$$
\left(\mathbb{C P}^{n}, J,\left[\nabla^{g_{F S}}\right]\right),
$$

where $\nabla^{g_{F S}}$ is the Levi-Civita connection of the Fubini-Study metric $g_{F S}$.

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- The $P$-invariant filtration $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{p}_{+}$, where $\mathfrak{p}_{+}$is the nilradical of $\mathfrak{p}$, induces a filtration on $\mathcal{A} M=\mathcal{G} \times p \mathfrak{g}$ by subbundles.

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- For a choice $\nabla \in[\nabla]$ the bundle $\mathcal{A} M=\mathcal{G} \times{ }_{p} \mathfrak{g}$ can be identified with

$$
T M \oplus \mathfrak{g l}(T M, J) \oplus T^{*} M
$$

and the Cartan curvature with a 2 -form on $M$ with values in:

$$
\mathfrak{g l}(T M, J) \oplus T^{*} M
$$

given by

$$
W^{\nabla}+C^{\nabla}
$$

where $W^{\nabla}$ is the c-projective Weyl curvature and $C^{\nabla}$ the c-projective Cotton-York tensor.

We have $W_{a b}^{\nabla c}{ }_{d}=R_{a b}^{\nabla{ }^{c}{ }_{d}}-\left(\partial \mathrm{P}^{\nabla}\right)_{a b}{ }^{c}{ }_{d}$, where

$$
\begin{gathered}
\left(\partial \mathrm{P}^{\nabla}\right)_{a b}{ }^{c}{ }_{d}:=\delta_{[a}{ }^{c}{ }^{P}{ }_{b] d}^{\nabla}-J_{[a}{ }^{c} \mathrm{P}_{b] e}^{\nabla} J_{d}{ }^{e}-\mathrm{P}_{[a b]}^{\nabla} \delta_{d}{ }^{c}-J_{[a}{ }^{e} \mathrm{P}_{b] e}^{\nabla} J_{d}{ }^{c}, \\
\mathrm{P}_{a b}^{\nabla}=\frac{1}{n+1}\left(R_{a b}^{\nabla}+\frac{1}{n-1}\left(R_{(a b)}^{\nabla}-J_{(a}{ }^{c} J_{b)}{ }^{d} R_{c d}^{\nabla}\right)\right),
\end{gathered}
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where $R_{a b}^{\nabla}:=R_{c a}^{\nabla c}{ }_{b}$ is the Ricci tensor of $\nabla$ and and $\mathrm{P}^{\nabla}$ the $c$-projective Schouten tensor of $\nabla$.

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- $2 n \geq 6$ : $W$ has two irreducible components (of complex type $(1,1)$ and $(2,0))$ and is independent of the choice of connection.
- $2 n=4: W$ is irreducible of complex type $(1,1)$; $W$ and the $(2,0)$-component of $C$ are independent of the choice of connection.
- If $W \equiv 0$ (respectively $W=0=C^{(2,0)}$ ), then $(M, J,[\nabla])$ is locally equivalent to $\left(\mathbb{C P}^{n}, J,\left[\nabla^{g_{F S}}\right]\right)$.


## 4. Metrisable c-projective structures

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For any section $\eta^{a b} \in \Gamma\left(S_{+}^{2} T M(-1)\right)$ and connection $\nabla \in[\nabla]$ the

$$
\text { trace-free-part }\left(\nabla_{a} \eta^{b c}\right)=: \operatorname{tfp}\left(\nabla_{a} \eta^{b c}\right)
$$

is independent of the choice of connection in [ $\nabla$ ]. Hence,

$$
D^{\mathrm{Met}}: \eta^{a b} \mapsto \operatorname{tfp}\left(\nabla_{a} \eta^{b c}\right)
$$

is a c-projectively invariant differential operator.

Mapping a Hermitian metric $g_{a b} \in \Gamma\left(S_{+}^{2} T^{*} M\right)$ on $(M, J)$ to $g^{a b} \mathrm{vol}(g)^{\frac{1}{n+1}}$ restricts to a bijection between compatible (pseudo-)Kähler metrics of ( $M, J,[\nabla]$ ) and non-degenerate solutions of $D^{\mathrm{Met}}(\eta)=0$.

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Let $\mathcal{T}$ be the tractor bundle associated to the representation $\mathbb{C}^{n+1}$ of $\operatorname{SL}(n+1, \mathbb{C})$ and set

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\mathcal{V}:=S_{+}^{2} \mathcal{T}=S_{+}^{2} T M(-1)+T M(-1)+\mathcal{E}(-1)
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## Prolongation

Elements in the kernel of $D^{\mathrm{Met}}$ are in bijective correspondence to sections that are parallel for a linear connection on $\mathcal{V}$. For a choice of connection $\nabla \in[\nabla]$, this connection is given by
$\nabla_{a}^{\text {prol }}\left(\begin{array}{c}\eta^{b c} \\ X^{b} \\ \rho\end{array}\right)=\left(\begin{array}{c}\left.\left.\nabla_{a} \eta^{b c}+\delta_{a}{ }^{(b} X^{c}\right)+J_{a}{ }^{(b} J_{d}{ }^{c}\right) X^{d} \\ \nabla_{a} X^{b}+\rho \delta_{a}{ }^{b}-\mathrm{P}_{a c} \eta^{b c} \\ \nabla_{a} \rho-\mathrm{P}_{a b} X^{b}\end{array}\right)+\frac{1}{n}\left(\begin{array}{c}0 \\ W_{a d}{ }^{b}{ }_{c} \eta^{d c} \\ -C_{a b c} \eta^{b c}\end{array}\right)$,

- The degree of mobility of $[\nabla]$ is defined to be the dimension of $\operatorname{ker} D^{\text {Met. }}$
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- Non-degenerate elements $\eta^{a b}$ in the kernel of $D^{\text {Met }}$ that satisfy $W_{a d}{ }^{b}{ }_{c} \eta^{d c}=0$ and $C_{a b c} \eta^{b c}=0$ corresponds to compatible (pseudo-)Kähler-Einstein metrics.
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- $S_{+}^{2} T M(-1)$ is isomorphic to $\wedge_{+}^{2} T M(-1)$ $\leadsto$ Hamiltonian 2-forms (Apostolov-Calderbank-Gauduchon).
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Another important c-projective invariant operator is the c-projective Hessian operator given by

$$
\begin{gathered}
D^{\mathrm{Hes}}: \mathcal{E}(1) \rightarrow S_{-}^{2} T^{*} M(1) \\
D^{\mathrm{Hes}} \sigma=\nabla_{(a} \nabla_{b)} \sigma+\mathrm{P}_{(a b)} \sigma-J_{(a}{ }^{c} J_{b)}{ }^{d}\left(\nabla_{c} \nabla_{d} \sigma+\mathrm{P}_{c d} \sigma\right)
\end{gathered}
$$

## Proposition

For a nowhere vanishing section $\sigma \in \Gamma(\mathcal{E}(1))$ we have $D^{\text {Hes }} \sigma=0$ $\Longleftrightarrow$ the Ricci tensor of the special connection corresponding to $\sigma$ (i.e. $\nabla \sigma=0$ ) is J-invariant.

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Observation
For any compatible (pseudo-)Kähler metric $g^{a b}=\eta^{a b}|\operatorname{det}(\eta)|$, the section $\operatorname{vol}(\mathrm{g})^{-\frac{1}{n+1}}=\operatorname{det}(\eta) \in \Gamma(\mathcal{E}(1))$ is in the kernel of the c-projective Hessian.
5. Consequences of degree of mobility $\geq 2$-Conserved quantities for geodesic flow

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## 5. Consequences of degree of mobility $\geq 2$-Conserved

 quantities for geodesic flowSuppose $(M, J, g)$ is a (pseudo-)Kähler manifold. Denote by $\Omega$ the associated Kähler form and by $\nabla$ its Levi-Civita connection, and consider the induced c-projective manifold ( $M, J,[\nabla]$ ).

## Proposition

Then $\sigma=h \operatorname{vol}(\mathrm{~g})^{-\frac{1}{n+1}} \in \Gamma(\mathcal{E}(1))$ is in the kernel of the c-projective Hessian

$$
K^{b}:=\Omega^{a b} \nabla_{a} h=J_{a}^{b} g^{a c} \nabla_{c} h,
$$

is a holomorphic Killing vector field with respect to $(J, g)$ (i.e. $h$ is a Killing potential or Hamilitonian for K).

Invariant Differential Pairing

$$
\begin{gathered}
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Then for any $s \in \mathbb{R}$ such that $\tilde{\eta}(s)$ is non-degenerate

$$
K(\tilde{\eta}(s), \tilde{\sigma}(t))=K(\tilde{\eta}(t)+(t-s) \eta, \tilde{\sigma}(t))=(t-s) K(\eta, \tilde{\sigma}(t))
$$

is a holomorphic Killing vector field with respect to the corresponding metric for all $t \in \mathbb{R}$.

- family of commuting Killing vector fields $K(t):=K(\eta, \tilde{\sigma}(t))$ corresponding to $(g, \tilde{g}) \leadsto$ family of Possion-commuting linear integrals $L_{t}$ for the geodesic flow
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- Integrability is related to the spectral theory of $A_{a}{ }^{b}=\operatorname{vol}(g)^{-\frac{1}{n+1}} \tilde{\eta}^{b c} g_{a c} \in \operatorname{End}(T M)$.

Theorem
Let $(M, J,[\nabla])$ be a c-projective manifold that admits compatible (pseudo-)Kähler metrics $g_{a b}$ and $\tilde{g}_{a b}$ associated to linearly independent solutions $\eta^{a b}$ and $\tilde{\eta}^{a c}=\eta^{a b} A_{b}{ }^{c}$ of the metrisability equation.

1. The number of functionally independent linear integrals $L_{s}$ of $g$ is equal to the number of nonconstant eigenvalues of $A$ at any generic point.
2. The number of functionally independent quadratic integrals $I_{t}$ of $g$ is equal to the degree of the minimal polynomial of $A$ at any generic point.
3. The integrals $I_{t}$ are functionally independent from the integrals $L_{s}$.
4. Consequences of degree of mobility $\geq 2$-Special connections on tractor bundles

## 6. Consequences of degree of mobility $\geq 2$-Special

 connections on tractor bundlesTheorem [Fedorova et al., 2014]
Suppose ( $M^{2 n}, J, g$ ) is a connected (pseudo-)Kähler manifold ( $2 n \geq 4$ ) of degree of mobility $\geq 3$. Then there exists constant $B \in \mathbb{R}$ such that all solutions of the metrisability equation $A^{a b} \in \Gamma\left(S_{+}^{2} T M\right)$ uniquely lift to sections of $\mathcal{V}$ that are parallel for the following connection

$$
\nabla_{a}\left(\begin{array}{c}
A^{b c} \\
\Lambda^{b} \\
\mu
\end{array}\right)=\left(\begin{array}{c}
\left.\left.\nabla_{a} A^{b c}+\delta_{a}{ }^{(b} \Lambda^{c}\right)+J_{a}{ }^{(b} J_{e}{ }^{c}\right) \Lambda^{e} \\
\nabla_{a} \Lambda^{b}+\mu \delta_{a}^{b}-2 B g_{a c} A^{b c} \\
\nabla_{a} \mu-2 B g_{a b} \Lambda^{b}
\end{array}\right),
$$

where $\nabla$ is the Levi-Civita connection of $g$ and densities are trivialised by $g$.

- this lift of $A^{a b}$ to a section of $\mathcal{V}$ differs from the lift that is parallel for the prolongation connection
- the formula for the connection occurs when one replaces $P_{a b}$ in the tractor connection by $2 B g_{a b}$
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Denote the Kähler curvature tensor of metrics of constant sectional holomorphic curvature 4 by:

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S_{a b c d}:=g_{a c} g_{b d}-g_{b c} g_{a d}+\Omega_{a c} \Omega_{b d}-\Omega_{b c} \Omega_{a d}+2 \Omega_{a b} \Omega_{c d}
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- At any point $x \in M$ there exist at most one constant $B \in \mathbb{R}$ such that

$$
G_{a b c d}^{B}:=R_{a b c d}-B S_{a b c d}
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has nullity at $x$, i.e. there exists $0 \neq v \in T_{x} M$ such that $G_{a b c d}^{B} v^{d}=0$.

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- We write $\mathcal{N}_{x}$ for $x \in M$ for the nullity distribution of $(M, J, g)$ and say that $g$ has nullity at $x$, if $\mathcal{N}_{x}$ is nonzero.


## Proposition

Suppose $\left(M^{2 n}, J, g\right)$ is a (pseudo-)Kähler manifold ( $2 n \geq 4$ ) and let $v^{d} \in T_{x} M$ be a nonzero tangent vector. Then the following statements are equivalent:

1. $v^{d} \in \mathcal{N}_{x}$
2. there exists a constant $B \in \mathbb{R}$ such that $W_{a b}{ }^{c}{ }_{d} v^{d}=\left(J_{a}{ }^{e} P_{e b}-2 B \Omega_{a b}\right) J_{d}{ }^{c} v^{d}$
3. $W_{a b}{ }^{c}{ }_{d} v^{a} v^{d}=0$ and $\left(W_{a b}{ }^{c}{ }_{d}+J_{b}{ }^{e} J_{f}{ }^{c} W_{a e}{ }^{f}{ }_{d}\right) v^{b}=0$.

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## Corollary

At any $x \in M$, the nullity distribution of $\mathcal{N}_{x}$ is a metric c-projective invariant, i.e. the same for c-projectively equivalent (pseudo-)Kähler metrics $g$ and $\tilde{g}$. Furthermore, if $\mathcal{N}_{x}$ is nonzero, then

$$
\widetilde{P}_{a b}-2 \widetilde{B} \tilde{g}_{a b}=\mathrm{P}_{a b}-2 B g_{a b}
$$

- In the above Theorem the constant $B$ is characterised by nullity unless all c-projectively equivalent metrics are affinely equivalent, since

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for any solution $A^{a b}$ is a solution of the metrisability equation.

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## Theorem

Let $(M, J, g)$ be a connected (pseudo-)Kähler manifold of degree of mobility at least 2. Assume that there is a dense open subset $U \subseteq M$ on which $(J, g)$ has c-projective nullity and denote by $B$ the corresponding function. Then the following hold:

- $B$ is constant
- any solution $A^{a b}$ of the metrisability equation lifts uniquely to a section of $\mathcal{V}$ which is parallel for the special tractor connection.


## Theorem (Yano-Obata conjecture)

Let $(M, g, J)$ be a complete connected Kähler manifold of real dimension $2 n \geq 4$. Then $\operatorname{Aff}_{0}(g, J)=\operatorname{CProj}_{0}(g, J)$, unless $(M, g, J)$ is actually compact and isometric to $\left(\mathbb{C P}^{n}, J, c g_{F S}\right)$ for some positive constant $c \in \mathbb{R}$.

In the compact case, this was first proved by Matveev-Rosemann resp. Bolsinov-Matveev-Rosemann in the (pseudo-)Kähler setting.

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- Then one can show that $g$ has nullity on a dense open subset with positive $B$ and hence $B$ is constant and any solution of the metrisability equation lifts to a section parallel for the special connection on $\mathcal{V}$ for that $B$
- For any solution $A^{a b} \in \Gamma\left(S_{+}^{2} T M\right)$ of the metrisability equation the function $\lambda=-\frac{1}{2} A_{a}{ }^{a}$ satisfies the following differential equation:

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\begin{aligned}
& \nabla_{a} \nabla_{b} \nabla_{c} \lambda=-B\left(2 \Lambda_{a} g_{b c}+g_{a b} \Lambda_{c}+g_{a c} \Lambda_{b}-\Omega_{a b} J_{c}{ }^{d} \Lambda_{d}-\Omega_{a c} J_{b}{ }^{d} \Lambda_{d}\right) \\
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Theorem (Tanno 1978)
Suppose a connected complete Kähler manifold $(M, J, g)$ admits a non-trivial solution $\lambda \in C^{\infty}(M)$ of the above equation for $B \in \mathbb{R}_{+}$. Then $(M, J, g) \cong\left(\mathbb{C P}^{n}, J, g_{F S}\right)$.

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Theorem (Matveev-N., 2017)
Suppose $(M, J, g)$ is a connected complete Kähler manifold of dimension $2 n \geq 4$ whose holomorphic sectional curvature is not a positive constant. Then the index of the subgroup $\operatorname{Aff}(J, g)$ in the group $\operatorname{CProj}(J, g)$ is at most 2.

