Integrability and Associativity

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Aim: Failure in integrability is closely related to failure in associativity.

Based on:

- joint work with Marius Crainic (Utrecht)
- PhD Thesis of my student Daan Michiels (UIUC)

Groupoids

Definition

A **groupoid** is a category where every arrow is invertible.

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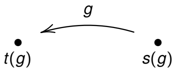
Source/target maps:

$$G = \{ \text{ arrows } \} \rightrightarrows \{ \text{ objects } \} = M$$



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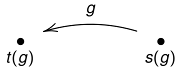
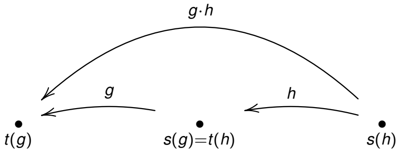
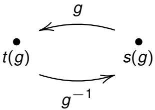


Multiplication, inversion and unit:





$m : G \,{}_s\!\times_t G \to G$ $\qquad \varepsilon : M \to G$ $\qquad i : G \longrightarrow G$

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Basic concepts:

- **source/target fibers** $s^{-1}(x)$, $t^{-1}(x)$ (closed submanifolds);
- **orbits:** $\mathcal{O}_x = t(s^{-1}(x))$ (immersed submanifolds);
- **isotropy Lie groups**: $G_x = s^{-1}(x) \cap t^{-1}(x)$ (Lie groups);

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There is an obvious notion of **(Lie) groupoid morphism**:

$$\begin{array}{ccc} G_2 & \xrightarrow{\ \Phi\ } & G_1 \\ \downarrow\downarrow & & \downarrow\downarrow \\ M_1 & \xrightarrow[\ \phi\ ]{} & M_2 \end{array}$$

Lie Groupoids and Moduli Spaces/Stacks

Given a Lie groupoid $G \rightrightarrows M$ its **coarse moduli space** is:

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Information is lost! Instead:

Definition

A **differentiable stack** is a Morita equivalence class of Lie groupoids.

(i.e, a Lie groupoid is an atlas for a stack and Morita equivalence is the notion of equivalence for such atlas) $\triangle$

Lie algebroids

Lie algebroids

Definition

A **Lie algebroid** is a vector bundle $A \to M$, with a Lie bracket $[\ ,\ ] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ and a bundle map $\rho : A \to TM$, called the anchor, such that:

$$[s_1, fs_2] = f[s_1, s_2] + \rho(s_1)(f)s_2, \quad \forall f \in C^\infty(M), s_1, s_2 \in \Gamma(A).$$

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△

Basic concepts:

- **Orbits:** $\rho([s_1, s_2]) = [\rho(s_1), \rho(s_2)] \Rightarrow \mathrm{Im}\,\rho$ is integrable (singular) distribution.
- **Isotropy Lie algebras:** For $x \in M$, $[\ ,\ ]$ restricts to Lie bracket on $\mathfrak{g}_x := \ker \rho$.

There is also a notion of **Lie algebroid morphism**.

Lie Functor

- Given $G \rightrightarrows M$ there is an **associated Lie algebroid** $A(G) \to M$:

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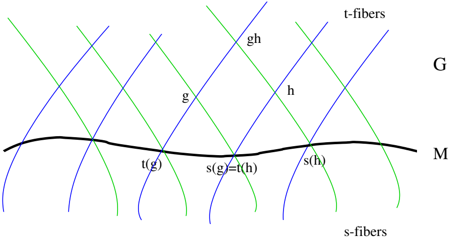

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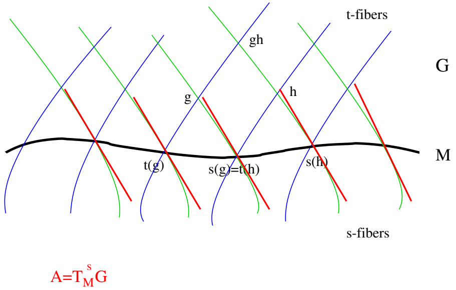

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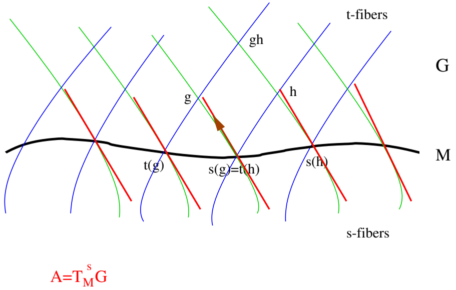

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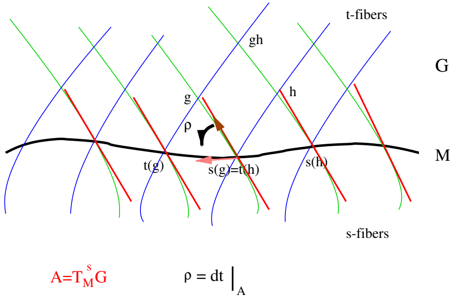

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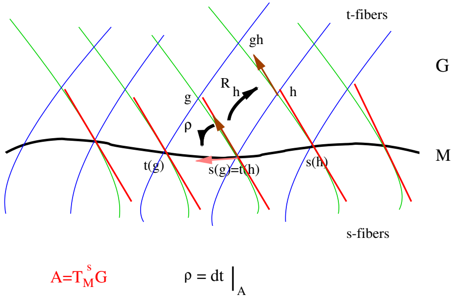

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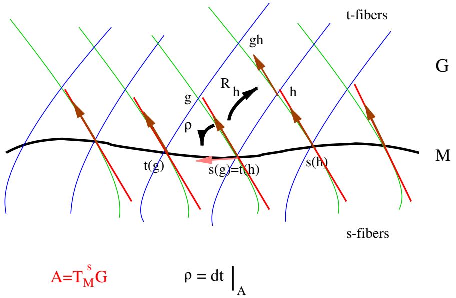

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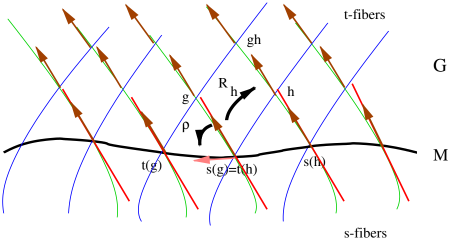

$A = T^s_M G$ $\qquad$ $\rho = dt|_A$ $\qquad$ $[s_1, s_2] = [X^{s_1}, X^{s_2}]|_M$

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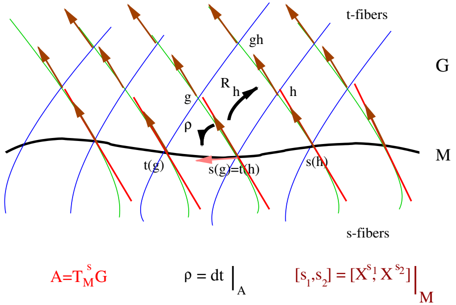

- Given a groupoid morphism $\Phi : G_1 \to G_2$ there is an **associated algebroid morphism** $\Phi_* : A(G_1) \to A(G_2)$. △

Lie's fundamental theorems

- **Lie I:** Given a source connected Lie groupoid $G \rightrightarrows M$ there is a unique source 1-connected Lie groupoid $\tilde{G} \rightrightarrows M$ with $A(\tilde{G}) \simeq A(G)$ and a unique cover of Lie groupoids $\tilde{G} \to G$;

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Lie III fails: not every Lie algebroid arises from a Lie groupoid. Obstructions are completely understood. $\triangle$

Why do we care?

Classification of Kähler metrics with vanishing Bochner tensor [Bryant 2001]:

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Find all principal U_n-bundles P equipped with 1-forms $\theta \in \Omega^1(P, \mathbb{C}^n)$, $\eta \in \Omega^1(P, \mathfrak{u}_n)$ and functions $(S, T, U) : P \to i\mathfrak{u}_n \oplus \mathbb{C}^n \oplus \mathbb{R}$ such that:

$$\begin{cases} \mathrm{d}\theta = -\eta \wedge \theta \\ \mathrm{d}\eta = -\eta \wedge \eta + S\theta^* \wedge \theta - S\theta \wedge \theta^* - \theta \wedge \theta^* S + (\theta^* \wedge S\theta) I_n \\ \mathrm{d}S = -\eta S + S\eta + T\theta^* + \theta T^* + \frac{1}{2}(T^*\theta + \theta^* T) I_n \\ \mathrm{d}T = -\eta T + (U I_n + S^2)\theta \\ \mathrm{d}U = T^* S\theta + \theta^* S T \end{cases}$$

Then $P = F_{\mathrm{U}_n}(M)$ is the U_n-structure of a Bochner-Kähler manifold, with tautological 1-form θ, connection 1-form η and invariants $\{S, T, U\}$.

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- **Solutions** can be found by integrating a Lie algebroid to a Lie groupoid!!
- The groupoid represents the **moduli space of Bochner-Kähler metrics**.

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Groupoids with an extra geometric structure (Riemannian groupoids, symplectic groupoids,...).

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Definition

A **symplectic groupoid** (G, Ω) is a Lie groupoid $G \rightrightarrows M$ with a symplectic form Ω which is multiplicative:

$$m^*\Omega = \mathrm{pr}_1^*\Omega + \mathrm{pr}_2^*\Omega, \quad m, \mathrm{pr}_1, \mathrm{pr}_2 : G\ {}_s\!\times_t G \to G.$$

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$$\Longleftarrow$$

$\triangle$

Monodromy

For a Lie algebroid $A \to M$ and $x \in M$ there is a **monodromy map**:

$$\partial_x : \pi_2(\mathcal{O}_x) \to G(\mathfrak{g}_x).$$

The image of this homomorphism is called the **monodromy group**:

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Theorem (Crainic & RLF)

A Lie algebroid $A \to M$ is integrable if and only if the monodromy groups $\mathcal{N}_x(A)$, $x \in M$, are uniformly discrete.

△

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Step 1 $G(A) = \frac{\{A\text{−}paths\}}{A\text{-homotopies}}$ (σ-model);

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Rmk: $G(\cdot)$ gives a functor:

$$\{\text{Lie Algebroids}\} \Longrightarrow \{\text{Topological Groupoids}\}.$$

Local Integrations

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- Multiplication is only defined on an open set:
$$(M\ {}_s\!\times_t\ G) \cup (G\ {}_s\!\times_t\ M) \subset \mathcal{U} \subset G\ {}_s\!\times_t\ G$$
- Inversion is only defined on an open set:
$$M \subset \mathcal{V} \subset G$$
- Associativity only holds on an open set:
$$(M_s\!\times_t G_s\!\times_t G)\cup(G_s\!\times_t M_s\!\times_t G)\cup(G_s\!\times_t G_s\!\times_t M) \subset \mathcal{W} \subset G_s\!\times_t G_s\!\times_t G$$

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Mal'cev Theorem for Lie Groupoids

G' is obtained by **restriction** from G if the arrows coincide ($G = G'$) and the domain of multiplications restrict $\mathcal{U}' \subset \mathcal{U}$.

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What about global associative (i.e., n-associative for all n)?

Theorem (Mal'cev, Michiels-RLF)

A local Lie groupoid is globally associative if and only if it is globalizable (i.e., a restriction of a global Lie groupoid) .

Corollary

A Lie algebroid is integrable if and only if it admits a local integration which is globally associative.

Associative Completion and Associators

Mal'cev's Theorem relies on the following construction for a local groupoid $G \rightrightarrows M$:

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- $W(G) = \bigcup_n \underbrace{G \ {}_s\times_t G \ {}_s\times_t \cdots \ {}_s\times_t G}_{n-\text{copies}}$ (well-formed words);
- $\sim$ equivalence relation on $W(G)$ generated by:

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_k) \sim (g_1, \ldots, g_i g_{i+1}, \ldots, g_k),$$

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Juxtaposition of words gives a (global) topological groupoid

$$\mathcal{AC}(G) = W(G)/\sim \rightrightarrows M \qquad \textbf{associative completion}$$

together with a morphism of (local) groupoids:

$$G \to \mathcal{AC}(G), \quad g \mapsto [(g)]. \qquad \textbf{completion map}$$

Associative Completion Functor

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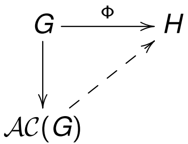
characterized by the **universal property**:

- For any (global) Lie groupoid $H \rightrightarrows M$ and any morphism of (local) Lie groupoids $\Phi : G \to H$ there exists a unique morphism of topological groupoids:



(i.e., $\mathcal{AC}(\cdot)$ is adjoint to the forgetful functor).

Associators

When is $\mathcal{AC}(G)$ a Lie groupoid?

$$\mathrm{Assoc}(G)_x = \{g \in G_x : (g) \sim (1_x)\} \quad \textbf{associators}$$

Theorem (Michiels-RLF)

$\mathcal{AC}(G)$ is a Lie groupoid iff $\mathrm{Assoc}(G)_x$ are uniformly discrete in G.

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Assume further that G is "shrinked" so that for all $x \in M$:

- G_x is 1-connected;
- $G_x \to G(\mathfrak{g}_x)$ is injective.

Theorem (Michiels-RLF)

If G is a shrunk local Lie groupoid with Lie algebroid A then:

$$\mathrm{Assoc}(G)_x = \mathcal{N}_x(A) \cap G_x.$$

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THANK YOU!