The ambient obstruction tensor and conformal holonomy

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Setting the scene

Conformal manifold: $(M, c = [g])$, $\quad \hat{g} \sim g \iff \exists f \in C^\infty(M) : \hat{g} = \mathrm{e}^{2f} g$

Invariant descriptions:

- (Normal conformal) Cartan-connection (and its holonomy group)
- Fefferman-Graham ambient metric (and the obstruction tensor)

Holonomy groups

- Let (E, ∇) be a vector bundle with connection ∇
  For example: $E = TM$, $\nabla = \nabla^g$ Levi-Civita connection of a metric g.
- $\mathcal{P}^\nabla_\gamma : E|_{\gamma(0)} \to E|_{\gamma(1)}$ parallel transport along a curve $\gamma : [0,1] \to M$
- $\mathrm{Hol}_p(E, \nabla) = \{\mathcal{P}^\nabla_\gamma \mid \gamma(0) = \gamma(1) = p\} \subset \mathbf{GL}(E|_p)$ holonomy group
  For example: $\mathrm{Hol}_p(E, \nabla^g) \subset \mathbf{O}(T_pM)$.
- For $\phi \in \Gamma(E)$ or $\phi \in \Gamma(\otimes^{r,s} E)$ with we have

$$\nabla\phi = 0 \quad \iff \quad \mathrm{Hol}_p(E, \nabla) \subset \mathrm{Stab}(\phi|_p)$$

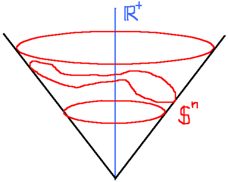
  For example: (M, g, J) Kähler, i.e., $\nabla^g J = 0 \iff \mathrm{Hol} \subset \mathbf{U}(\frac{n}{2})$

Both descriptions of conformal geometry have associated holonomy groups.

Flat model of conformal geometry

Light cone in Minkowski space:

$$\mathcal{C} := \left\{(r_-, \mathbf{x}, r_+) \in \mathbb{R}^{1,n+1} \setminus \{0\} \mid 2r_+r_- + \|\mathbf{x}\|^2 = 0,\ r_+ > 0\right\} \circlearrowleft \mathbf{SO}^0(1, n+1)$$



$$\begin{array}{rcl} \iota \ : \ \mathbb{R}^{n+1} \times \mathbb{R}^+ & \longrightarrow & \mathbb{R}^{1,n+1} \\ (\mathbf{y} = (y^0, \ldots y^n), t) & \mapsto & t \cdot \left(\frac{y^0-1}{\sqrt{2}}, y^1, \ldots, y^n, \frac{y^0+1}{\sqrt{2}}\right), \end{array}$$

$\Longrightarrow \iota(\mathbb{S}^n \times \mathbb{R}^+) = \mathcal{C}$ and $\mathbb{S}^n = \mathbb{P}\mathcal{C} = \mathbf{SO}(1, n+1)/P$, where $P = \mathrm{Stab}(\mathbb{R} \cdot s_-)$ stabiliser of the null line $(r_-, \mathbf{0}, 0)$ in $\mathcal{C}$.

The flat Minkowski-metric

$$g = 2dr_+dr_- + d\mathbf{x} \cdot d\mathbf{x},$$

with $r_\pm = \frac{t}{\sqrt{2}}(y^0 \pm 1)$ and $x^i = ty^i$ becomes

$$\iota^* g \ = \underbrace{(\|\mathbf{y}\|^2 - 1)}_{=0 \text{ on } \mathbb{S}^n} dt^2 + 2t \underbrace{(y \cdot d\mathbf{y})}_{=0 \text{ on } \mathbb{S}^n} dt + \ t^2 \underbrace{(d\mathbf{y} \cdot d\mathbf{y})}_{\text{round metric on } \mathbb{S}^n} \overset{\rho = \frac{\|\mathbf{y}\|^2 - 1}{2}}{=} 2\rho\, dt^2 + 2t\, d\rho\, dt + \underbrace{t^2 g_{round}}_{\in [g_{round}]}$$

and induces a conformal structure on $\mathbb{S}^n \rightsquigarrow$ $Conf(\mathbb{S}^n, g_r) = \mathbf{SO}(1, n+1)$.

The Fefferman-Graham ambient metric

- (M, g) semi-Riemannian manifold of dim n.
- A *pre-ambient metric (in normal form) for* (M, g) is a metric $\widetilde{g}$ on an open nbhd $\widetilde{M}$ of $\{1\} \times M \times \{0\}$ in $\mathbb{R}^+ \times M \times \mathbb{R} = \{(t, x, \rho)\}$ of the form

$$\widetilde{g} = 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2\, g_\rho$$

with a ρ-dependent family of metrics g_ρ on M such that

$$g_\rho|_{\rho=0} = g.$$

- A pre-ambient metric $\widetilde{g}$ for g is an *ambient metric* if

$$\widetilde{Ric} = O(\rho^\infty), \qquad \text{when } n \text{ is odd} \qquad (1)$$

$$\left.\begin{array}{rcl} \widetilde{Ric} & = & O(\rho^{\frac{n}{2}-1}) \\ \mathrm{tr}_g(\iota^* \widetilde{Ric}) & = & O(\rho^{\frac{n}{2}}) \end{array}\right\} \qquad \text{when } n \text{ is even,} \qquad (2)$$

where $\iota : M \to \widetilde{M}$ is the injection $x \mapsto (1, x, 0)$ and $\widetilde{Ric}$ is the Ricci tensor of $\widetilde{g}$.
- Conformally invariant: If g and $\hat{g} = \mathrm{e}^{2f} g$ are conformally equivalent, $\widetilde{g}$ ambient metric for g, then there is a diffeomorphism ψ such that $\psi^* \widetilde{g}$ is an ambient metic for $\hat{g}$.

Existence & uniqueness of ambient metrics [Fefferman-Graham '87 & '07]

If an ambient metric exits and is unique it provides an invariant description of conformal structures: conformal invariants then are just Riemannian invariants of the ambient metric.

If *n* is odd:

- $\exists$ an ambient metric, and it is unique in the following sense:
  for ambient metrics $\widetilde{g}_1$ and $\widetilde{g}_2$ there is a diffeom $\Phi : \widetilde{M}_1 \to \widetilde{M}_2$: with $\Phi|_{\rho=0} = Id$ and $\widetilde{g}_1 - \Phi^*\widetilde{g}_2 = O(\rho^\infty)$.
- If g is analytic, there is a unique analytic ambient metric $\widetilde{g}$ with $Ric(\widetilde{g}) = 0$.

Existence & uniqueness of ambient metrics [Fefferman-Graham '87 & '07]

If $n = 2s$ is even:

- $\exists$ an ambient metric, unique in the sense that for ambient metrics $\widetilde{g}_1$ and $\widetilde{g}_2$ there is a diffeom $\Phi : \widetilde{M}_1 \to \widetilde{M}_2$: with $\Phi|_{\rho=0} = Id$ and $\widetilde{g}_1 - \Phi^*\widetilde{g}_2 = O(\rho^s)$.
- $\exists$ conformally covariant, symmetric $(0,2)$-tensor $\mathcal{O}$, the *Fefferman-Graham obstruction tensor*, with $\mathrm{tr}(\mathcal{O}) = 0$ and $\mathrm{div}(\mathcal{O}) = 0$ and such that $Ric(\widetilde{g}) = O(\rho^\infty)$ implies $\mathcal{O} = 0$. It is

$$\iota^*\widetilde{Ric} = c_n \rho^{s-1}\mathcal{O} + \rho^s T, \quad c_n = constant, \; T = tensor\ on\ M$$

- Even if $\mathcal{O} = 0$, ambient metrics are not unique to all orders.
- $\mathcal{O} = \Delta^{s-2}\left(\Delta \mathrm{P} - \nabla^2 \mathrm{tr}(\mathrm{P})\right) +$ lower order terms, where

$$\mathrm{P} = \tfrac{1}{n-2} Ric(g) - \tfrac{scal}{2(n-1)} g,$$

  is the Schouten tensor and Δ is the Laplacian of g.
- If $n = 4$, then $\mathcal{O}$ is the Bach tensor $B_{ij} = \nabla^k\nabla_k \mathrm{P}_{ij} - \nabla^k\nabla_i \mathrm{P}_{kj} - \mathrm{P}^{kl} W_{kijl}$.

Examples of conformal structures with explicit ambient metrics

- An ambient metric with $\widetilde{Ric} = 0$ always exists if $[g]$ contains an Einstein metric: if $g_\Lambda \in [g]$ an Einstein-metric with $\mathsf{P} = \Lambda g$, then

$$
\begin{aligned}
\widetilde{g} &= 2\rho dt^2 + 2td\rho dt + t^2 \left(1+\rho\Lambda\right)^2 g_\Lambda \\
&= \begin{cases} -\frac{1}{2\Lambda}\mathrm{d}s^2 + \underbrace{\frac{1}{2\Lambda}\mathrm{d}r^2 + r^2 g_\Lambda}_{\text{cone metric}}, & \text{if } \Lambda \neq 0, \quad (r = t(1+\Lambda\rho), s = t(1-\Lambda\rho)) \\ \\ 2\mathrm{d}v\mathrm{d}t + t^2 g_0, & \text{if } \Lambda = 0, \quad (v = t\rho). \end{cases}
\end{aligned}
$$

  In both cases, $(\widetilde{M}, \widetilde{g})$ admits a parallel vector field, $\frac{\partial}{\partial s}$ or $\frac{\partial}{\partial v}$.
- Conversely, if the ambient metric exists and admits a parallel vector field of length λ, then on an open dense set, locally $[g]$ contains an Einstein metric g_Λ with $\Lambda = -\lambda$.

Examples of conformal structures with explicit ambient metrics

- Conformal structures defined by $(2,3,5)$ distributions in dimension 5 [Nurowski '05].
  In general they are not conformally Einstein, but in some cases one can find explicit ambient metrics [Nurowski/L '12]
- Conformal structure defined by generic rank 3 distributions on $\mathbb{R}^6$ [Bryant '06]. In general neither conformally Einstein nor $\mathcal{O} = 0$, but there are examples with explicit $\widetilde{Ric} = 0$ ambient metrics [Anderson/Nurowski/L,'15]

The normal conformal Cartan connection

Recall the flat model of conformal geometry

$$\mathbb{S}^{p,q} = \mathbf{SO}^0(p+1, q+1)/P, \quad \text{where } P = \text{stabiliser of a null line } L.$$

The Maurer-Cartan form of $\mathbf{SO}^0(p+1,q+1) \to \mathbf{SO}^0(p+1,q+1)/P$ generalises to the *normal conformal Cartan connection* ω on a P-bundle $\mathcal{P}$ over $(M,[g])$ with values in the $|1|$-graded Lie algebra

$$\mathfrak{so}(p+1,q+1) = \mathfrak{g}_{-1} \oplus \underbrace{\overset{\mathfrak{co}(p,q)}{\overset{\|}{\mathfrak{g}_0}} \oplus \mathfrak{g}_1}_{= \mathfrak{p} = \mathfrak{co}(p,q) \ltimes \mathbb{R}^{p,q} = \mathfrak{stab}(L)}$$

[Recall Andreas' or Katharina's talks]

The Cartan connection extends to a principle fibre bundle connection ω on $\tilde{\mathcal{P}} = \mathcal{P} \times_P \mathbf{SO}^0(p,q)$.

The conformal tractor bundle and the normal conformal tractor connection

- *Tractor bundle* $\mathbb{T} = \mathcal{P} \times_P \mathbb{R}^{p+1,q+1} = \tilde{\mathcal{P}} \times_{\mathbf{SO}(p+1,q+1)} \mathbb{R}^{p+1,q+1}$ with induced metric $\overline{g}$.
- $\mathbb{T}$ is filtered as $\mathcal{L} \subset \mathcal{L}^\perp \subset \mathbb{T}$, where

$$\mathcal{L} = \mathcal{P} \times_P L, \qquad \mathcal{L}^\perp = \mathcal{P} \times_P L^\perp.$$

  (Recall: $P = \mathrm{Stab}(L)$.)
- ω defines a covariant derivative $\overline{\nabla}$ on $\mathbb{T}$.
- $\overline{\nabla}$ admits a parallel section $\iff$ locally on an open dense set $[g]$ contains an Einstein metric.
- The *conformal holonomy* at $p \in M$

$$\mathrm{Hol}_p(M, [g]) := \left\{ \mathcal{P}_\gamma^{\overline{\nabla}} \in \mathbf{O}(\mathbb{T}_p) \mid \gamma(0) = \gamma(1) = x \right\}$$

  with Lie algebra $\mathfrak{hol}(M, [g]) \subset \mathfrak{so}(p+1, q+1)$.

The tractor bundle and tractor connection

Any metric $g \in [g]$ gives a reduction of $\mathcal{P}$ to the bundle of orthonormal frames $\mathcal{O}^g$ with respect to the injection $\mathbf{SO}(p,q) \subset P$. The $\mathbf{SO}(p,q)$-invariant splitting $\mathbb{R}^{p+1,q+1} = \mathbb{R}e_- \oplus^\perp \mathbb{R}^{p,q} \oplus^\perp \mathbb{R}e_+$ with $L = \mathbb{R}e_-$ yields two sections s_- and s_+ of $\mathbb{T}$ and a splitting

$$\mathbb{T} = \mathcal{P} \times_P \mathbb{R}^{p+1,q+1} = \mathcal{O}^g \times_{\mathbf{SO}(p,q)} \mathbb{R}^{p+1,q+1} = \mathbb{R}s_- \oplus^\perp \underbrace{(\mathcal{O}^g \times_{\mathbf{SO}(p,q)} \mathbb{R}^{p,q})}_{= s_-^\perp \cap s_+^\perp = TM} \oplus^\perp \mathbb{R}s_+$$

with the normal conformal Tractor connection given by

$$\overline{\nabla}_X s_- = X, \qquad \overline{\nabla}_X Y = \nabla_X Y - \mathsf{P}(X,Y)s_- - g(X,Y)s_+, \qquad \overline{\nabla}_X s_+ = \mathsf{P}(X)^\sharp.$$

Changing the metric to $\hat{g} = \mathrm{e}^{2\sigma} g$ yields

$$\hat{s}_- = \mathrm{e}^{-\sigma} s_-, \quad \hat{X} = \mathrm{e}^{-\sigma}(X + d\sigma(X)s_-), \qquad \hat{s}_+ = \mathrm{e}^{\sigma} s_+ - \mathrm{e}^{-\sigma}\left(\frac{\|\nabla\sigma\|^2}{2} s_- - \nabla\sigma\right)$$

The identification

$$\mathcal{L}^\perp / \mathcal{L} = \mathcal{P} \times_P (L^\perp / L) = \mathcal{O}^g \times_{\mathbf{SO}(p,q)} \mathbb{R}^{p,q} \simeq TM$$

is independent of the of the chosen metric g.

The conformal holonomy distribution [Lischewski-L '15]

Recall that $\mathfrak{hol}([g]) = \mathfrak{hol}(\overline{\nabla})$ is contained in

$$\mathfrak{so}(p+1,q+1) = \mathfrak{g}_{-1} \oplus \underbrace{\mathfrak{g}_0 \oplus \mathfrak{g}_1}_{=\mathfrak{p}} = \left\{ \begin{pmatrix} a & v & 0 \\ u & X & v^\sharp \\ 0 & u^\flat & -a \end{pmatrix} \middle| \begin{array}{l} (a,X) \in \mathfrak{co}(p,q) \\ u \in \mathbb{R}^{p,q}, v \in (\mathbb{R}^{p,q})^* \end{array} \right\}.$$

Using $\mathrm{pr} : \mathcal{L}^\perp \to \mathcal{L}^\perp/\mathcal{L} \simeq TM$, define *the conformal holonomy distribution* in TM,

$$\mathcal{E} := \mathfrak{hol}([g]) \cap \mathfrak{g}_1 = \left\{ \mathrm{pr}(\mathrm{im}(A)) \mid A \in \mathfrak{hol}([g]), A\mathcal{L} = 0, A\mathcal{L}^\perp \subset \mathcal{L} \right\} \subset TM.$$

- There is an open dense subset M_0 in M such that $rk(\mathcal{E})$ is constant over the connected components of M_0, *"$\mathcal{E}$-adapted sets"*.

Proposition

Over each $\mathcal{E}$-adapted set we have that:

- *$\mathcal{E}$ is totally null (i.e. light-like) or $\mathfrak{hol}([g]) = \mathfrak{so}(p+1,q+1)$.*
- *$\mathcal{E}$ is either integrable or one of the generic distributions in dimension 5 or 6 that define the two exceptional conformal structures.*

The proofs use that s_- behaves like an Euler vector field $\overline{\nabla}_X s_- = X$ and that $\nabla_X H \in \mathfrak{hol}$ if $H \in \mathfrak{hol}$.

- $V \in \mathcal{E}$ can be identified with $s_- \wedge V = \begin{pmatrix} 0 & V^\flat & 0 \\ 0 & 0 & -V \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{hol}$
- Then for $X \in TM$ we get
$$\overline{\nabla}_X(s_- \wedge V) = X \wedge V + \nabla_X V^\flat \wedge s_+ - g(X,V) s_- \wedge s_+ = \begin{pmatrix} -g(X,V) & \nabla_X V^\flat & 0 \\ 0 & X \wedge V & -\nabla_X V \\ 0 & 0 & g(X,V) \end{pmatrix}$$
is in $\mathfrak{hol}$.
- Take Lie brackets
$$\left[s_- \wedge \hat{V}, \overline{\nabla}_X(s_- \wedge V)\right] = \begin{pmatrix} 0 & g(X,V)\hat{V}^\flat + g(X,\hat{V})V^\flat - g(\hat{V},V)X^\flat & 0 \\ 0 & 0 & \vdots \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{hol}$$
- Take $V = \hat{V}$ and $X \in V^\perp$, then $g(V,V) \neq 0$, implies $\mathcal{E} = \mathbb{R}V \oplus V^\perp = TM$.
- $\left[\overline{\nabla}_X(s_- \wedge \hat{V}), \overline{\nabla}_X(s_- \wedge V)\right] = \begin{pmatrix} 0 & \dots & 0 \\ 0 & g(X,X) V \wedge \hat{V} & \vdots \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{hol}$ shows that $\mathfrak{p} \subset \mathfrak{hol}$
- Differentiate this again gives $\mathfrak{g}_{-1} \subset \mathfrak{hol}$ and hence $\mathfrak{hol} = \mathfrak{so}(p+1, q+1)$
- Integrability: similar computations give
$$[V, \hat{V}] - \frac{2g(\nabla_X V, \hat{V})}{g(X,X)} X \in \mathcal{E}, \qquad \forall\, X \in (V, \hat{V})^\perp,\ g(X,X) \neq 0$$
...

Conformal holonomy and the obstruction tensor [Lischewski-L ’15]

Theorem

Let $(M,[g])$ be a conformal manifold, of even dim n and signature (p,q). Then

- *the image of $\mathcal{O}$ when considered as an endomorphism field on M is contained in $\mathcal{E} = \mathfrak{hol}([g]) \cap \mathfrak{g}_1$ and in particular, $\mathrm{rk}(\mathcal{O}) \leq \dim(\mathfrak{hol}([g]) \cap \mathfrak{g}_1)$.*
- *If $\mathfrak{hol}([g]) \neq \mathfrak{so}(p+1,q+1)$, then the image of $\mathcal{O}$ is totally null and in particular, $\mathrm{rk}(\mathcal{O}) \leq \min(p,q)$.*
  I.e., if $\mathrm{rk}(\mathcal{O}) > \min(p,q)$, then $\mathfrak{hol}([g]) = \mathfrak{so}(p+1,q+1)$.

Corollary

If $(M,[g])$ be a Riemannian conformal manifold with $\mathfrak{hol}([g]) \neq \mathfrak{so}(1,n+1)$, then $\mathcal{O} = 0$.

Idea of the proof:

1. Relation between the tractor bundle and the ambient metric:
   - Affine bundle isomorphism of $(T\widetilde{M}|_M, \widetilde{\nabla})$ and $(\mathbb{T} = \mathbb{R}s_- \oplus TM \oplus \mathbb{R}s_+, \overline{\nabla})$ via

$$\partial_t \mapsto s_-, \qquad TM \overset{Id}{\mapsto} TM, \qquad \partial_\rho \mapsto s_+$$

   - Hence, $\mathrm{Hol}(M,[g]) \subset \mathrm{Hol}(\widetilde{M},\widetilde{g})$.
   - Moreover, one can show that [Čap, Gover, Graham & Hammerl '15]

$$\mathfrak{hol}(\mathbb{T},\overline{\nabla}) = \mathfrak{hol}^{\frac{n}{2}-1}(\widetilde{M},\widetilde{g}) := \left\{ \begin{array}{l} (\widetilde{\nabla}_{X_k} \ldots \widetilde{\nabla}_{X_3}\widetilde{R})(X_2,X_1) \mid X_i \in T\widetilde{M}, \\ X_i \notin TM \oplus \mathbb{R}\partial_t \text{ for at most } \frac{n-2}{2} \text{ many } X_i \end{array} \right\},$$

2. Relation between the obstruction tensor and conformal holonomy:
   - Ambrose-Singer holonomy theorem implies

$$\widetilde{\nabla}_{X_1} \cdots \widetilde{\nabla}_{X_k} \widetilde{R}(Y,Z) \in \mathfrak{hol}(\widetilde{M},\widetilde{g})$$

   - Use $\mathcal{O} = \widetilde{\nabla}_{\partial_\rho}^{\frac{n-2}{2}} \widetilde{Ric}|_{\rho=0}$ to express $\mathcal{O}$ in terms of derivatives of ambient curvature,

$$\partial_t \wedge \mathcal{O}(X,.) \in \mathfrak{hol}^{\frac{n}{2}-1}(\widetilde{M},\widetilde{g}) = \mathfrak{hol}(\mathbb{T},\overline{\nabla}).$$

Consequences

With $\mathrm{im}(\mathcal{O}) \subset \mathcal{E} = \mathfrak{hol}([g]) \cap \mathfrak{g}_1$, any conformal holonomy reduction, i.e., $\mathfrak{hol}([g]) \subsetneq \mathfrak{so}(p+1, q+1)$, imposes conditions on $\mathrm{im}(\mathcal{O})$:

- Let $\overline{\alpha} \in \Lambda^{k+1}\mathbb{T}^*$ a $\overline{\nabla}$-parallel form, $\overline{\alpha} = s_+^\flat \wedge \alpha + \dots$ with $\alpha \in \Lambda^k T^*M$ the associated *normal conformal Killing form*. Then
$$\mathrm{im}(\mathcal{O}) \wedge \alpha = 0.$$
- If $\overline{\mathcal{N}} \subset \mathbb{T}$ is a $\mathfrak{hol}([g])$-invariant subspace, then $\overline{\mathcal{N}} = \mathbb{R}s_+ \oplus \mathcal{N}$ with $\mathcal{N} \subset \mathcal{L}^\perp$ and
$$\mathrm{im}(\mathcal{O}) \subset N = \mathrm{pr}(\mathcal{N}) \subset TM.$$
In this situation, locally in an open dense set there is a metric $g \in [g]$ such that:
  - If N is non degenerate, then $g = g_1 \times g_2$ is a *special Einstein product*, i.e., a product of Einstein metrics with $\Lambda_1 = -\Lambda_2$, [Armstrong '05, Leitner '05]
  - If N is totally null, then N is parallel for ∇^g and $\mathrm{im}(Ric^g) \subset N$, [L '06, Nurowski-L '11, Lischewski '15]

Special conformal structures

Let $(M, [g])$ be a conformal manifold of dimension $n = p + q$

$\overline{\nabla}$-parallel	$(M, [g])$	$\mathrm{Hol}([g]) \subset$	$rk(\mathcal{O})$
$\alpha \in \mathbb{T}^*$	conf Einstein	$\begin{pmatrix} 1 & * \\ 0 & H \end{pmatrix}$	0
$\alpha \in \Lambda^{p+1}\mathbb{T}^*$ decomp.	special Einstein product	$\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$	0
	Parallel null p-plane $\mathcal{N}$	$\begin{pmatrix} H_1 & * \\ 0 & H_2 \end{pmatrix}$	$Im(\mathcal{O}) \subset \mathcal{N}$
$J \in \Lambda^2\mathbb{T}^*$, $J^2 = -1$	normal conformal vf Fefferman space	$\mathbf{SU}(\frac{p}{2} + 1, \frac{q}{2} + 1)$	≤ 1 0 if $p = 1$, $\subsetneq$
$\alpha \in \Lambda^3\mathbb{T}^*$, $n = 5$	$(2, 3, 5)$ distribution [Nurowski]	$\mathbf{G}_{2(2)}$	—
$\alpha \in \Lambda^4\mathbb{T}^*$, $n = 6$	$(3, 6)$ distribution [Bryant]	$\mathbf{Spin}(3, 4)$	≤ 1 if $\subsetneq$

The obstruction tensor vanishes in each of the following cases:

1. $[g]$ is Riemannian and $\mathfrak{hol}([g]) \neq \mathfrak{so}(1, n+1)$;
   i.e., $\mathcal{O}$ also obstructs the existence of parallel tractors that do not come from the tractor metric;
2. $[g]$ is Lorentzian and $\mathfrak{hol}([g]) \subsetneq \mathfrak{su}(1, n/2)$;
3. if $\mathfrak{hol}([g])$ is reducible, i.e., locally in a dense open set in M the conformal class contains an Einstein metric or special Einstein product [Gover/Leitner09].
4. $\exists$ a non-null normal conformal vector field V or 2 normal conformal vector fields (e.g. Fefferman spaces over quaternionic contact structures in signature $(4k+3, 4l+3)$ — if $\mathfrak{hol}([g]) \subset \mathfrak{sp}(k+1, l+1)$),
5. $\exists$ twistor spinors $\varphi_{i=1,2}$ such that $\{X \in TM \mid X \cdot \varphi_i = 0\}$ are complementary.

$\mathrm{rk}(\mathcal{O}) \leq 1$ for each of the following cases:

1. $(p, q) = (3, 3)$ and $\mathfrak{hol}([g]) \subsetneq \mathfrak{spin}(3, 4)$ (Bryant's conformal structures) ☹
2. $(p, q) = (n, n)$ and $\mathfrak{hol}([g]) \subset \mathfrak{gl}(n + 1) \subset \mathfrak{so}(n + 1, n + 1)$;
3. $\exists$ normal conformal vector field (e.g. Fefferman conformal structures, i.e., $\mathfrak{hol}([g]) \subset \mathfrak{su}(r + 1, s + 1)$);
4. the action of $\mathrm{Hol}(M, [g])$ on the light cone $\mathcal{C} \subset \mathbb{R}^{p+1,q+1}$ does not have an open orbit.

For each of these geometries one can give an explicit subspace $V \subset TM$ with $\mathrm{Im}(\mathcal{O}) \subset V$ at each point.

Thank you!