Spin^c-quantisation

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I Spin^c-Dirac operators

Dirac operators

Idea (Dirac, 1928):

- Special relativity: $E^2 = p^2c^2 + m^2c^4$
- Quantum mechanics: $p^2 = \hbar^2\Delta$, with

$$\Delta = -\sum_j \frac{\partial^2}{\partial x_j^2}$$

  the Laplacian.
- So look for **Dirac operator** D representing E, such that

$$D^2 = \Delta + \text{zero-order terms}.$$

Clifford bundles

Tool to construct Dirac operators: **Clifford bundles**.

- If V is a (fin. dim., real) vector space, with a quadratic form q, its **Clifford algebra** is

$$\mathrm{Cl}(V,q) := \frac{\bigoplus_{j=0}^{\infty} V^{\otimes j}}{\text{ideal generated by } \{v \otimes v + q(v); v \in V\}}.$$

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- If M is a manifold with a Riemannian metric B, we have the **Clifford bundle** $\mathrm{Cl}(TM, B) \to M$ whose fibre at $m \in M$ is $\mathrm{Cl}(T_m M, B_m)$.

Spin^c-structures

Let (M, B) be an oriented, even-dimensional Riemannian manifold.

A Spin^c-**structure** on M is a Hermitian vector bundle $\mathcal{S} \to M$ with an isomorphism of complex algebra bundles

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This exists if and only if the second Stiefel–Whitney class $w_2(M) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ is the image of a class in $H^2(M; \mathbb{Z})$. Examples:

- all manifolds of dimension ≤ 4;
- (stably) almost complex manifolds, hence all symplectic manifolds;
- Spin-manifolds ($w_2(M) = 0$).

Spin^c-Dirac operators

Let $\mathcal{S} \to M$ be a Spin^c-structure. A **Clifford connection** on $\mathcal{S}$ is a Hermitian connection $\nabla^{\mathcal{S}}$ such that

$$\nabla^{\mathcal{S}}_v c(w)s = c(\nabla^{LC}_v w)s + c(w)\nabla^{\mathcal{S}}_v s \quad \in \Gamma^\infty(\mathcal{S}).$$

for all $s \in \Gamma^\infty(\mathcal{S})$ and $v, w \in \mathrm{Vect}(M)$. Here ∇^{LC} is the Levi–Civita connection on TM. Clifford connections always exist.

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Definition

The Spin^c-**Dirac operator** D associated to a Clifford connection $\nabla^{\mathcal{S}}$ is

$$\boxed{D : \Gamma^\infty(\mathcal{S}) \xrightarrow{\nabla^{\mathcal{S}}} \Gamma^\infty(T^*M \otimes \mathcal{S}) \xrightarrow{c} \Gamma^\infty(\mathcal{S}).}$$

Properties

- We indeed have (Bochner, 1949)

$$D^2 = \Delta + R,$$

where $\Delta := (\nabla^{\mathcal{S}})^*\nabla^{\mathcal{S}}$ and $R \in \operatorname{End}\mathcal{S}$.

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- The Dirac operator D is formally self-adjoint. (Essentially s.a. if M is complete.)
- Since M is even-dimensional, there is a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, and D interchanges $\mathcal{S}^+$ and $\mathcal{S}^-$. Let $D^\pm$ be the restriction

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  Then $(D^+)^* = D^-$.
- If M is **compact**, then $\ker D$ is **finite-dimensional**. (Since D is elliptic.)

The index of a Spin^c-Dirac operator

From now on,

- M is a compact, connected, oriented, even-dimensional Riemannian manifold;
- $\mathcal{S} \to M$ is a Spin^c-structure;
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We saw that $\ker D$ is finite-dimensional, and the adjoint of

$$D^+ : \Gamma^\infty(\mathcal{S}^+) \to \Gamma^\infty(\mathcal{S}^-).$$

is D^-. Hence we have the index of D^+,

$$\boxed{\operatorname{index} D^+ = \dim\ker D^+ - \dim\ker D^-} \quad \in \mathbb{Z}.$$

It is independent of $\nabla^{\mathcal{S}}$.

Example: complex manifolds

If M has an almost complex structure, it has a Spin^c-structure

$$\mathcal{S} := \bigwedge_{\mathbb{C}} TM \cong \bigwedge^{0,*} T^*M,$$

with, for all $v \in TM$, $\alpha \in \bigwedge_{\mathbb{C}} TM$,

$$c(v)\alpha = v \wedge \alpha - v^* \lrcorner \alpha.$$

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If M is complex and L is holomorphic:

$$\operatorname{index} D^+ = \sum_{j \text{ even}} \dim H^j_{\mathrm{Dolb}}(M; L) - \sum_{j \text{ odd}} \dim H^j_{\mathrm{Dolb}}(M; L),$$

the **Riemann–Roch number** of $L \to M$.

Equivariant indices

Let G be a compact, connected Lie group, acting isometrically on M. Suppose the action lifts to $\mathcal{S}$. (There is a criterion in equivariant cohomology for this.) Suppose $\nabla^{\mathcal{S}}$ is G-invariant.

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Then $\ker D$ is a finite-dimensional representation of G, so we have the **equivariant index**

$$\boxed{\mathrm{index}_G\, D^+ = [\ker D^+] - [\ker D^-]}$$

in the representation ring

$$R(G) = \{[V] - [W]; V, W \text{ finite-dimensional representations}\}.$$

Representation theory

Equivariant indices of Spin^c-Dirac operators are a rich source of representations.

Example

Let $T < G$ be maximal torus. The manifold $M = G/T$ is complex. Take

$$\mathcal{S}_L = \bigwedge^{0,*} T^*M \otimes L,$$

with $L := G \times_T \mathbb{C}_\xi$, where T acts on $\mathbb{C}_\xi$ with weight $\xi \in \hat{T} \subset i\mathfrak{t}^*$.

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$$\boxed{\operatorname{index}_G D^+ = \pm[\pi_{\xi'}^*],}$$

where $\pi_{\xi'}$ is the irreducible representation with highest weight ξ', obtained from ξ via the Weyl group action.

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where $\pi_{\xi'}$ is the irreducible representation with highest weight ξ', obtained from ξ via the Weyl group action.

Central question: How does $\operatorname{index}_G D^+$ decompose into irreducible representations in general?

II Geometric quantisation

Quantisation and reduction

Idea from physics:

$$\begin{tikzcd}[column sep=huge, row sep=large]
\boxed{\begin{array}{c}\text{Classical}\\ \text{mechanics}\\ \text{with symmetry}\end{array}} \arrow[r, "\text{quantisation}", "Q"'] \arrow[d, "\text{reduction } R"', "\text{(use symmetry to simplify)}"] & \boxed{\begin{array}{c}\text{Quantum}\\ \text{mechanics}\\ \text{with symmetry}\end{array}} \arrow[d, "R"] \\
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Quantisation commutes with reduction:

$$Q \circ R = R \circ Q, \qquad \text{or} \qquad \text{‘}[Q, R] = 0\text{’}.$$

Milestones

The quantisation commutes with reduction principle was stated rigorously and proved for **compact** symmetry groups and **compact** classical phase spaces M.

1982	Guillemin–Sternberg	M Kähler
1998, 1999	Meinrenken, Sjamaar	M symplectic
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The original problem was directly motivated by physics, but especially the Spin^c-case has much wider relevance, to geometry, representation theory and index theory.

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$$Q(M, \omega) := L^2(M, \omega^n/n!)$$

($\dim M = 2n$). But:

- $L^2(M)$ is **too large**, e.g. we would like $Q(\mathbb{R}^2, dp \wedge dq) = L^2(\mathbb{R})$.
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Initial idea: define

$$Q(M, \omega) \subset \Gamma^\infty(L),$$

for a line bundle $L \to M$ with $c_1(L) = [\omega] \in H^2(M)$. (Under an integrality assumption on $[\omega]$.)

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Then quantise $f \in C^\infty(M)$ as

$$Q(f) := \nabla^L_{X_f} - 2\pi i f,$$

where $(\nabla^L)^2 = 2\pi i \omega$, if this operator on $\Gamma^\infty(L)$ preserves $Q(M, \omega)$.

Geometric quantisation: definitions

1. If (M, ω) is Kähler, we take $Q(M, \omega)$ to be the **holomorphic sections** of a holomorphic line bundle $L \to M$ with $c_1(L) = [\omega]$:

$$Q(M, \omega) := H^0_{\mathrm{Dolb}}(M; L).$$

Up to isomorphism, this is determined by $\dim H^0_{\mathrm{Dolb}}(M; L)$.

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2. One can also include other Dolbeault cohomology groups:

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If $\kappa^* \otimes L$ is positive, this reduces to the first definition by Kodaira's vanishing theorem. (With κ the canonical line bundle.)
3. We saw that the second definition equals
$$Q(M, \omega) = \mathrm{index}\, D^+,$$
the **index of a** Spin^c**-Dirac operator**. This can still be defined if (M, ω) is not Kähler, or even if M just has a Spin^c-structure.

Equivariant geometric quantisation

Suppose a compact, connected Lie group G acts on M, preserving ω.

Bott: define the **geometric quantisation** of the action as

$$\boxed{Q_G(M,\omega) := \mathrm{index}_G\, D^+} \quad \in R(G),$$

for the Spin^c-structure $\bigwedge^{0,*} T^*M \otimes L$ on M, w.r.t. a G-invariant almost complex structure J such that $\omega(-, J-)$ is a Riemannian metric.

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This definition of geometric quantisation as an index is the right one for applications to representation theory. (E.g. Borel–Weil–Bott.)

This is physics-inspired maths.

Quantum reduction

Quantum reduction at an irreducible representation π of G is the map

$$R_\pi : R(G) \to \mathbb{Z}$$

defined by

$$\boxed{R_\pi([V]) = [V : \pi],}$$

the multiplicity of π in a representation V of G.

Moment maps

Suppose there is a **moment map**

$$\mu : M \to \mathfrak{g}^*,$$

which is equivariant, and satisfies

$$\boxed{d\langle \mu, X \rangle = -\omega(X^M, -)} \quad \in \Omega^1(M).$$

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If the action preserves $h \in C^\infty(M)$, then

$$v_h(\langle \mu, X\rangle) = 0$$

for all $X \in \mathfrak{g}$, where v_h is the Hamiltonian vector field of f. So μ is **conserved** by the dynamics determined by h.

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Example

Consider n particles in $\mathbb{R}^3$. Then $M = \mathbb{R}^{6n}$, and

- μ is total linear momentum for the translation action by $\mathbb{R}^3$;
- μ is total angular momentum for the rotation action by SO(3).

Classical reduction

Let $\xi \in \mathfrak{g}^*$ be a regular value of μ. The **reduced space** at ξ is

$$\boxed{M_\xi := \mu^{-1}(\xi)/G_\xi,}$$

where G_ξ is the stabiliser of ξ w.r.t. the coadjoint action. This is an orbifold.

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Theorem (Marsden–Weinstein, 1974)

The symplectic form ω descends to a symplectic form ω_ξ on M_ξ.

Quantisation commutes with reduction

$$\begin{tikzcd} G \circlearrowright (M,\omega) \arrow[rr, "Q_G"] \arrow[d, "R_\xi"'] & & \mathrm{index}_G(D^+) \arrow[d, "R_\pi"] \\ (M_\xi,\omega_\xi) \arrow[r, "Q"] & \mathrm{index}\, D^+_{M_\xi} & {[\mathrm{index}_G\, D^+ : \pi]} \end{tikzcd}$$

Theorem (Meinrenken, 1998)

If ξ is the highest weight of π, then

$$[\mathrm{index}_G\, D^+ : \pi] = Q(M_\xi,\omega_\xi) := \mathrm{index}\, D^+_{M_\xi} \quad \in \mathbb{Z}.$$

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If ξ is the highest weight of π, then

$$[\mathrm{index}_G\, D^+ : \pi] = Q(M_\xi, \omega_\xi) := \mathrm{index}\, D^+_{M_\xi} \quad \in \mathbb{Z}.$$

This is a **localisation** result, more refined than fixed point formulas.

Meinrenken–Sjamaar (1999) generalised this to singular reduced spaces. Meinrenken's proof is geometric in nature. Tian–Zhang (1998) gave an analytic proof, and Paradan (2001) gave a topological proof.

Example: branching rules for SU(2)

Irreducible representations of SU(2):

$$\pi_n = Q_{\mathrm{SU}(2)}(S^2, \omega_n)$$

for $n = 0, 1, 2, 3, \ldots$.

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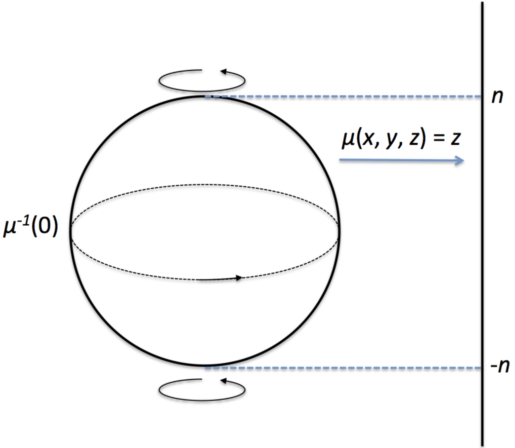
for $n = 0, 1, 2, 3, \ldots$. $\mathrm{U}(1) \hookrightarrow \mathrm{SU}(2)$ acts on S^2 by double rotations.



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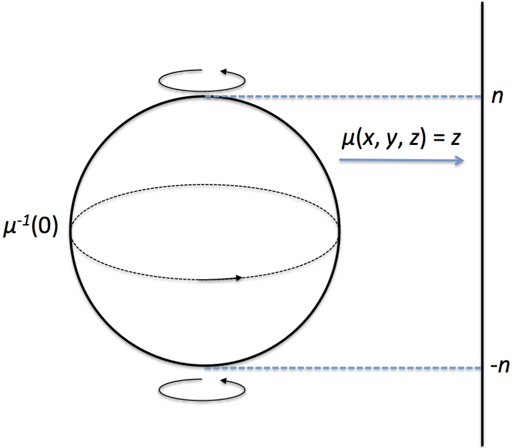

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$[Q, R] = 0$ implies

$$\begin{aligned} \pi_n|_{\mathrm{U}(1)} = Q_{\mathrm{U}(1)}(S^2, \omega_n) = \bigoplus_{l \in \mathbb{Z}} Q(\mu^{-1}(l)/\,\mathrm{U}(1))\mathbb{C}_l \\ = \mathbb{C}_{-n} \oplus \mathbb{C}_{-n+2} \oplus \cdots \oplus \mathbb{C}_{n-2} \oplus \mathbb{C}_n. \end{aligned}$$

Here $\mathbb{C}_l$ is the representation of U(1) in $\mathbb{C}$ with weight $l \in \mathbb{Z}$.

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$[Q, R] = 0$ implies

$$\pi_n|_{\mathrm{U}(1)} = Q_{\mathrm{U}(1)}(S^2, \omega_n) = \bigoplus_{l \in \mathbb{Z}} Q(\mu^{-1}(l)/\,\mathrm{U}(1))\mathbb{C}_l$$
$$= \mathbb{C}_{-n} \oplus \mathbb{C}_{-n+2} \oplus \cdots \oplus \mathbb{C}_{n-2} \oplus \mathbb{C}_n.$$

Here $\mathbb{C}_l$ is the representation of U(1) in $\mathbb{C}$ with weight $l \in \mathbb{Z}$.

This goes in steps of 2 because we use orbifold indices.

Branching rules and $[Q, R] = 0$

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This is good because

- It links physics, geometry and representation theory together.
- In particular, geometry of μ reflects the classical limit of a multiplicity function. (See also: Duistermaat–Heckman measure $\mu_*(\omega^n/n!)$.)
- We can apply geometry to representation theory. E.g.
  - The image of the moment map determines which representations occur.
  - If a reduced space is a point, the multiplicity is 0 or 1.

III The Spin^c-case

Spin^c-manifolds

Now consider the more general case where M is a compact, connected, even-dimensional manifold with a Spin^c-structure $\mathcal{S} \to M$. Suppose the action by G lifts to $\mathcal{S}$. Then one can still define

$$Q_G(M, \mathcal{S}) := \operatorname{index}_G D^+ \quad \in R(G),$$

although the relation with physics is lost in the Spin^c-context.

Spin^c-moment maps

The **determinant line bundle** of the Spin^c-structure $\mathcal{S}$ is

$$\lambda := \mathrm{Hom}_{\mathrm{Cl}(TM,B)}(\overline{\mathcal{S}}, \mathcal{S}) \to M,$$

where $\overline{\mathcal{S}}$ equals $\mathcal{S}$ with the opposite complex structure. Let ∇^λ be a G-invariant Hermitian connection on λ.

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The Spin^c-**moment map** of ∇^λ is the map $\mu : M \to \mathfrak{g}^*$ defined by

$$\boxed{2\pi i \langle \mu, X \rangle = \nabla^\lambda_{X^M} - \mathcal{L}_X} \quad \in \mathrm{End}(\lambda) = C^\infty(M, \mathbb{C}).$$

Here $\mathcal{L}_X$ is the Lie derivative of sections of λ. If $2\pi i \omega := (\nabla^\lambda)^2$ is symplectic, this reduces to the previous definition of moment maps. (Kostant's formula.)

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One can still define the reduced space

$$M_\xi := \mu^{-1}(\xi)/G_\xi$$

for a regular value $\xi \in \mathfrak{g}^*$ of μ. This inherits a Spin^c-structure $\mathcal{S}_\xi \to M_\xi$ from $\mathcal{S} \to M$ (constructed by Paradan–Vergne).

Spin^c-quantisation commutes with reduction

Theorem (Paradan–Vergne, 2014)

If the action by G on M has abelian stabilisers, then for all irreducible representations π of G,

$$[\mathrm{index}_G\, D^+ : \pi] = Q(M_\xi, \mathcal{S}_\xi) := \mathrm{index}\, D^+_{M_\xi} \quad \in \mathbb{Z},$$

if ξ is the highest weight of π plus half the sum of a positive root system. In general, $[\mathrm{index}_G\, D^+ : \pi]$ is a finite sum of quantisations of reduced spaces.

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This is now a result on the general index theory of Spin^c-Dirac operators, not just on geometric quantisation.

Corollary (Originally Atiyah–Hirzebruch, 1970)

If M has a Spin*-structure to which the action lifts, and the action on M is not trivial, then* $\mathrm{index}_G\, D^+ = 0$. *In particular,* $\int_M \hat{A}(M) = 0$.

Noncompact manifolds and groups

It is a natural question if $[Q, R] = 0$ extends to the noncompact case.

- In **physics**, many classical phase spaces are noncompact, e.g. cotangent bundles.
- In **mathematics**, representation theory of noncompact groups is much more complicated than for compact groups, and $[Q, R] = 0$ can shed light on this.

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- In **mathematics**, representation theory of noncompact groups is much more complicated than for compact groups, and $[Q, R] = 0$ can shed light on this.

There are some challenges for noncompact manifolds and groups.

- If M **is noncompact**, then $\ker D$ is infinite-dimensional in general.
- If G **is noncompact**, then it is not clear what kind of object the quantisation should be. (E.g. $R(G)$ is no longer well-defined.)

So the first question is how to generalise the equivariant index to the noncompact case.

Results in the noncompact case

	M symplectic	M Spin^c
M/G compact	H.–Landsman (2008) Mathai–Zhang (2010) H. (2010, 2015)	H.–Mathai (2014)
M noncompact G compact	Ma–Zhang (2014) Paradan (2011) H.–Song (2017)	H.–Song (2017)
M/G, G noncompact	H.–Mathai (2014)	H.–Mathai (2014) H.-Song (to appear)

- The results for M/G compact involve K-theory of C^*-algebras.
- The results for M noncompact but G compact are stated in terms of several equivalent analytic and topological indices.

IV Application to representation theory

Realising representations

Let G be a semisimple Lie group. Let π be an irreducible representation of G that is a direct summand of $L^2(G)$ (assuming it exists). Let $K < G$ be maximal compact and $T < K$ a maximal torus.

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Theorem (Paradan, 2003)

The restriction $\pi|_K$ is the K-equivariant Spin^c-**quantisation** *of an elliptic* **coadjoint orbit** *of G,*

$$\begin{aligned} \pi|_K &= Q_K(\mathcal{O}); \\ \mathcal{O} &= \mathrm{Ad}^*(G)\xi \cong G/T. \end{aligned}$$

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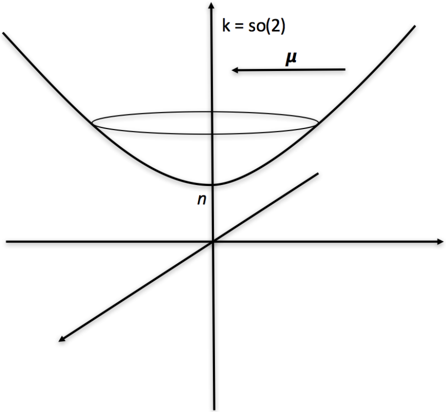
Important: $\mathcal{O}$ has to be treated as a Spin^c-manifold, not as a symplectic manifold. Because

- the natural complex structure on $\mathcal{O}$ is not compatible with the symplectic form;
- the natural line bundle on $\mathcal{O}$ is not a prequantum line bundle.

Example: $G = \mathrm{SL}(2, \mathbb{R})$

Then $\pi = D_n^{\pm}$ for $n = 1, 2, 3, \ldots$, the **discrete series** of $\mathrm{SL}(2, \mathbb{R})$. And $D_n^{+}|_K$ is the K-equivariant Spin^c-quantisation of



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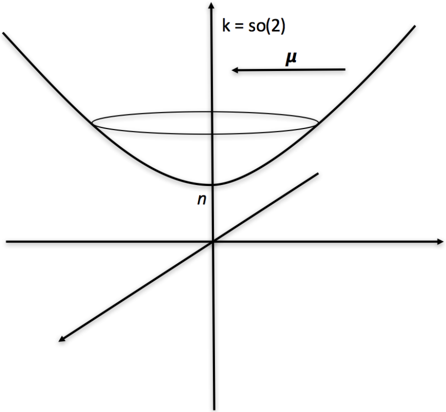

By $[Q, R] = 0$ (Paradan, H.-Song),

$$D_n^+|_K = \bigoplus_{l \in \mathbb{Z}} Q(\mu^{-1}(l)/\mathrm{SO}(2))\mathbb{C}_l$$
$$= \mathbb{C}_{n+1} \oplus \mathbb{C}_{n+3} \oplus \mathbb{C}_{n+5} \oplus \cdots$$

Realising representations (2)

Theorem (H.–Song–Yu, 2016)

For any irreducible representation π of G occurring in $L^2(G)$ (not necessarily discretely), $\pi|_K$ is the Spin^c*-quantisation of a coadjoint orbit.*

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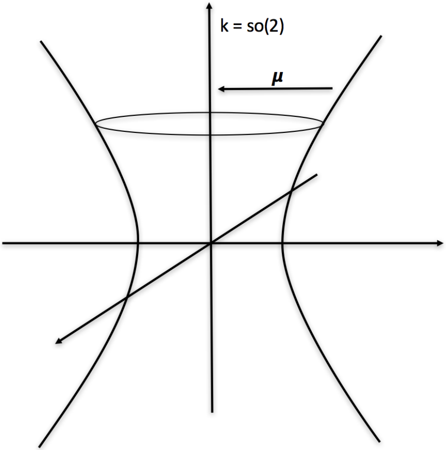
Corollary

Geometric sufficient condition for representations to occur with multiplicities 0 *or* 1*.*

Example: the principal series of $\mathrm{SL}(2,\mathbb{R})$

Principal series: $\pi = P^{\pm}_{i\nu}$ for $\nu \geq 0$. $P^{+}_{i\nu}|_K$ is the Spin^c-quantisation of



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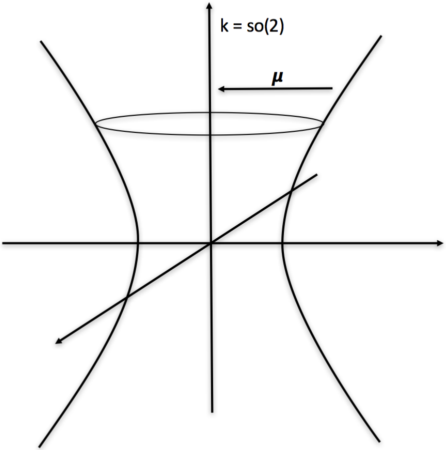
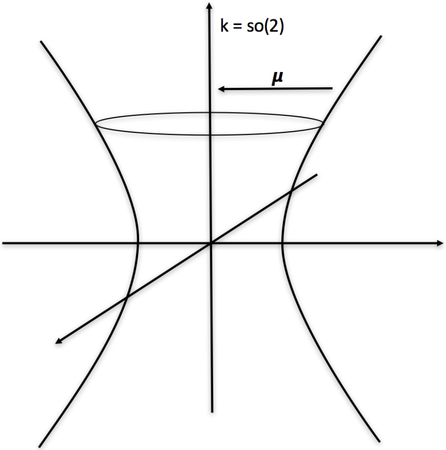


By $[Q, R] = 0$,

$$P^{+}_{i\nu}|_K =$$
$$\cdots \oplus \mathbb{C}_{-2} \oplus \mathbb{C}_0 \oplus \mathbb{C}_2 \oplus \mathbb{C}_4 \oplus \cdots$$

Example: the principal series of $SL(2, \mathbb{R})$ (2)

$P^{-}_{i\nu}|_K$ is the Spin^c-quantisation with a different Spin^c-structure of



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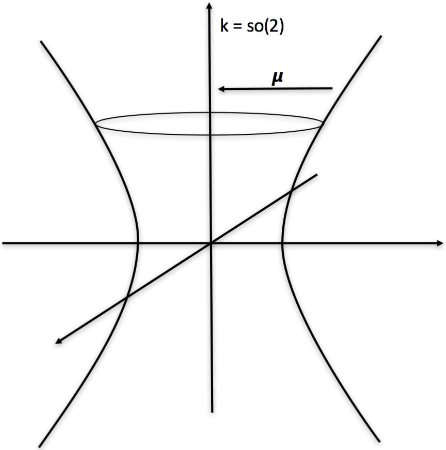
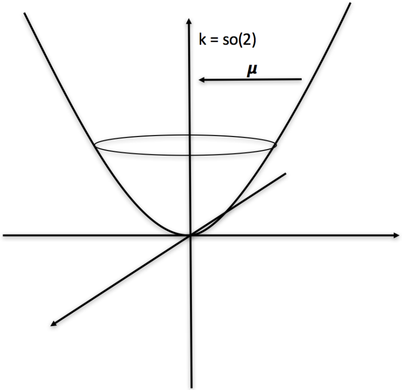


By $[Q, R] = 0$,

$$P^-_{i\nu}|_K = \cdots \mathbb{C}_{-3} \oplus \mathbb{C}_{-1} \oplus \mathbb{C}_1 \oplus \mathbb{C}_3 \oplus \cdots$$

Limits of discrete series of $\mathrm{SL}(2, \mathbb{R})$

The other representations occurring in $L^2(G)$ are the **limits of discrete series** $\pi = D_0^{\pm}$. For D_0^{+}, the map μ is now a shifted and deformed version of the the projection map from a coadjoint orbit.



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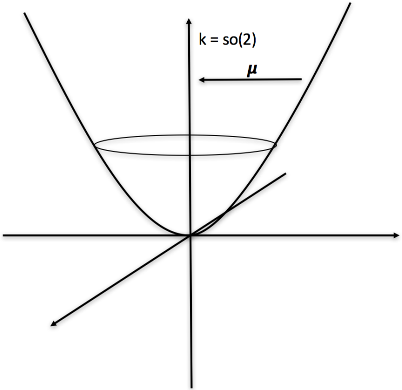

Similarly to the discrete series case, we obtain

$$D_0^+|_K = \mathbb{C}_1 \oplus \mathbb{C}_3 \oplus \mathbb{C}_5 \oplus \cdots \quad \text{and} \quad D_0^-|_K = \mathbb{C}_{-1} \oplus \mathbb{C}_{-3} \oplus \mathbb{C}_{-5} \oplus \cdots$$

Thank you