Calculus Structure on the Variational Complex

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1. Lie conformal algebras
   - Motivation from QFT
   - Definition

2. Cochain and chain complexes
   - Definitions
   - Calculus structure

3. Variational complex
   - Construction using Lie conformal algebras
   - Relevance to integrable hierarchies
Quantum fields in 2 dimensions

- Quantum fields are operator-valued distributions.

- **Basic problem:** For $x, y \in \mathbb{R}^d$,

  \[
  \lim_{x \to y} \phi(x) \psi(y) = ?
  \]

- When $d = 2$, introduce

  \[
  z = x^0 - x^1, \quad \bar{z} = x^0 + x^1 \quad \Rightarrow \quad |x|^2 = z \bar{z}
  \]

- Quantum fields are elements in $\text{End}(\mathcal{H})[[z, z^{-1}, \bar{z}, \bar{z}^{-1}]].$

- **Locality axiom:**

  \[
  (z - w)(\bar{z} - \bar{w}) < 0 \quad \Rightarrow \quad [\phi(z, \bar{z}), \psi(w, \bar{w})] = 0
  \]
Operator product expansion

- Chiral quantum fields: \( \partial \bar{z} \phi(z, \bar{z}) = 0. \)

- Locality axiom for chiral quantum fields:
  \[
  (z - w)^N[\phi(z), \psi(w)] = 0, \quad N \gg 0
  \]

- Operator product expansion:
  \[
  \phi(z)\psi(w) = \sum_{j=0}^{N-1} \frac{\phi(w)(j)\psi(w)}{(z - w)^{j+1}} + :\phi(z)\psi(w): \]
  
  where
  \[
  \phi(w)(j)\psi(w) = \text{Res}_z (z - w)^j[\phi(z), \psi(w)]
  \]
\( \lambda \)-bracket

- The \( j^{th} \)-products satisfy the Borcherds identity:

\[
\sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} \left( \phi_{m+n-j} \psi_{(k+j)\chi} - (-1)^n \psi_{(k+n-j)} \phi_{(m+j)\chi} \right)
\]

\[
= \sum_{j \in \mathbb{Z}_+} \binom{m}{j} \phi_{(n+j)\psi}(m+k-j)\chi
\]

for all \( m, n, k \in \mathbb{Z} \) and quantum fields \( \phi, \psi \) and \( \chi \).

- Generating function:

\[
[\phi \lambda \psi] = \sum_{j \in \mathbb{Z}_+} (\phi_{(j)\psi}) \frac{\lambda^j}{j!}
\]
A Lie conformal algebra is a $\mathbb{C}[\partial]$-module $A$ with a $\lambda$-bracket $[\cdot,\cdot] : A \otimes A \to \mathbb{C}[\lambda] \otimes A$, satisfying

- **Sesquilinearity**
  
  \[
  [\partial a_\lambda b] = -\lambda [a_\lambda b] \\
  [a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]
  \]

- **Skewsymmetry**
  
  \[
  [a_\lambda b] = -[b_{-\partial -\lambda} a]
  \]

- **Jacobi identity**
  
  \[
  [a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]
  \]

**Note:** $[\cdot,\cdot]|_{\lambda=0}$ defines a Lie bracket on $A/\partial A$. 
A module over a Lie conformal algebra $A$ is a $\mathbb{C}[\partial]$-module $M$ endowed with a $\lambda$-action $A \otimes M \rightarrow \mathbb{C}[\lambda] \otimes M$, denoted $a \otimes m \mapsto a_{\lambda} m$, such that

- $(\partial a)_{\lambda} m = -\lambda a_{\lambda} m$
- $a_{\lambda}(\partial m) = (\partial + \lambda)a_{\lambda} m$
- $a_{\lambda}(b_{\mu} m) - b_{\mu}(a_{\lambda} m) = [a_{\lambda} b]_{\lambda + \mu} m$
Example: Virasoro conformal algebra

- Centerless Virasoro algebra
  
  \[ [L_m, L_n] = (m - n)L_{m+n} \]

- Quantum fields: \( L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \)
  
  \[
  L(z)L(w) = \frac{2L(w)}{(z - w)^2} + \frac{\partial L(w)}{(z - w)} + :L(z)L(w): 
  \]

- Associated Lie conformal algebra:
  
  \( \text{Vir} = \mathbb{C}[\partial]L, \quad [L_L L] = (\partial + 2\lambda)L \)

- All finite irreducible non-trivial Vir-modules are
  
  \( M_{\Delta, \alpha} = \mathbb{C}[\partial]v, \quad L_\lambda v = (\partial + \alpha + \Delta \lambda)v \)

  where \( \Delta \in \mathbb{C}^*, \alpha \in \mathbb{C} \). (Cheng-Kac, 1997)
Lie conformal algebra complex

- No Weyl complete reducibility theorem or Levi decomposition.

- (Bakalov-Kac-Voronov, 1999)

\[ \Gamma^\bullet(A, M) = \tilde{\Gamma}^\bullet(A, M) / \partial \tilde{\Gamma}^\bullet(A, M) \]

- **Drawbacks:**
  - torsion information lost,
  - too small when \( A \neq \text{Tor}(A) \oplus (\mathbb{C}[\partial] \otimes U) \).
Let \( C_-[\lambda_1, \ldots, \lambda_k] \) denote the space of polynomials with the \( C[\partial] \)-module structure:

\[
\partial P(\lambda_1, \ldots, \lambda_k) = -(\lambda_1 + \cdots + \lambda_k)P(\lambda_1, \ldots, \lambda_k)
\]

The space of \( k \)-cochains \( C^k(A, M) \) is the space of \( C \)-linear maps

\[
c : A^\otimes k \rightarrow C_-[\lambda_1, \ldots, \lambda_k] \otimes_{C[\partial]} M
\]

\[
a_1 \otimes \cdots \otimes a_k \mapsto c_{\lambda_1, \ldots, \lambda_k}(a_1, \cdots, a_k)
\]

satisfying certain skewsymmetry and sesquilinearity relations.
Example: Virasoro conformal algebra cohomology

- For finite Lie conformal algebras $A$:

$$C^\bullet(A, M) \cong \frac{\Delta^\bullet(\text{Lie}_- A, M)}{\partial \Delta^\bullet(\text{Lie}_- A, M)}$$

- $\text{Lie}_- Vir = \text{Vect}(\mathbb{C})$.

- Non-trivial cohomology only when $\Delta = 1 - \frac{(3r^2 \pm r)}{2}$ and $\alpha = 0$:

$$\dim H^q(Vir, M_{\Delta, \alpha}) = \begin{cases} 
2 & \text{for } q = r + 1 \\
1 & \text{for } q = r \text{ or } r + 2 
\end{cases}$$
Chain space

- Let $\mathcal{M}_k \subset M[[x_1, \ldots, x_k]]$ denote the subspace of formal power series, such that
  $$\partial \phi(x_1, \ldots, x_k) = (\partial x_1 + \cdots + \partial x_k)\phi(x_1, \ldots, x_k)$$

- The space of $k$-chains $C_k(A, M)$ is the quotient of $A^\otimes k \otimes \mathcal{M}_k$ by the relations
  - $a_1 \otimes \cdots \partial a_i \cdots \otimes a_k \otimes \phi = -a_1 \otimes \cdots \otimes a_k \otimes (\partial x_i \phi)$
  - $\sigma(a_1 \otimes \cdots \otimes a_k \otimes \phi) = \text{sign}(\sigma)a_1 \otimes \cdots \otimes a_k \otimes \phi, \quad \sigma \in S_k$

- $C_\bullet(A, M)$ is not a complex in general.
Calculus structure

- $C_\bullet(A, M)$ is a Gerstenhaber algebra.

- Bracket on 1-chains:
  
  1. The pair $(A, M)$ is a Lie conformal algebroid.
  
  2. Extends to a Lie conformal algebroid $(M \otimes A, M)$ with $\lambda$-bracket
     
     $$[(m \otimes a)_\lambda(n \otimes b)]' = ((e^{\partial M \varphi \lambda} m)n) \otimes [a_\lambda b]$$

  3. Bracket on $C_1(A, M) \cong (A \otimes M)/\partial(A \otimes M)$ is $[\cdot, \cdot]'|_{\lambda=0}$.

- Contraction operator and Lie derivative provide a representation of $C_\bullet$ on the complex $(C^\bullet, d)$. 
Calculus of variations

- **Basic problem:** Minimize
  \[
  S = \int_a^b f \left( u_i, u'_i, u''_i, \ldots, u^{(n)}_i \right) \, dx, \quad u_i \in \mathcal{V}, \ i \in I = \{1, \ldots, N\}
  \]

- **Euler-Lagrange equations:**
  \[
  \frac{\delta f}{\delta u_i} := \sum_{k=0}^{n} (-\partial)^k \frac{\partial f}{\partial u_i^{(k)}} = 0, \quad i \in I
  \]

- **Variational derivative:** \( \frac{\delta}{\delta u} : \mathcal{V} \to \mathcal{V}^N \).

- **Basic identity:**
  \[
  \left[ \frac{\partial}{\partial u^{(n)}_i}, \partial \right] = \frac{\partial}{\partial u^{(n-1)}_i} \quad \Rightarrow \quad \partial \mathcal{V} \subset \text{Ker} \frac{\delta}{\delta u}
  \]
Variational complex

- (Helmholtz) $F \in \operatorname{Im} \frac{\delta}{\delta u} \Rightarrow D_F(\partial) = D_F(\partial)^*$ where

$$D_F(\partial)_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial F_i}{\partial u_j^{(n)}} \partial^n$$

- Restatement:

$$\mathcal{V}/\partial \mathcal{V} \xrightarrow{\frac{\delta}{\delta u}} \mathcal{V}^N \xrightarrow{D(\partial) - D(\partial)^*} \Omega^2(\mathcal{V}) \rightarrow \ldots$$

- **Question**: what are the higher cochains and differentials?
Construction of the variational complex

- Define $A = \bigoplus_{i \in I} \mathbb{C}[\partial] u_i$ with trivial $\lambda$-bracket.

- Define a representation of $A$ on $\mathcal{V}$ by

$$u_{i\lambda} f = \sum_{n \in \mathbb{Z}_+} \lambda^n \frac{\partial f}{\partial u_i^{(n)}}, \quad i \in I$$

and extend by sesquilinearity.

- The variational complex is $(C^\bullet(A, \mathcal{V}), d)$. 
Higher cochains and differentials

- For $k \geq 2$, elements in $C^k(A, \mathcal{V})$ are skewsymmetric $k$-differential operators $S : (\mathcal{V}^N)^k \to \mathcal{V}/\partial \mathcal{V}$ of the form

$$S(P^1, \ldots, P^k) = \int \sum_{i_r \in I, n_r \in \mathbb{Z}_+} f_{i_1, \ldots, i_k}^{n_1, \ldots, n_k} (\partial^{n_1} P^1_{i_1}) \cdots (\partial^{n_k} P^k_{i_k})$$

- The differential is

$$(dS)(P^1, \ldots, P^{k+1}) = \sum_{s=1}^{k+1} (-1)^{s+1} (X_{P^s} S)(P^1, \ldots, P^{k+1})$$

- $C_\bullet(A, \mathcal{V})$ is the Gerstenhaber algebra of evolutionary polyvector fields.
Construction of integrable hierarchies

• Bi-hamiltonian system: $\frac{du}{dt} = \{\int h_1, u\}_0 = \{\int h_0, u\}_1$.

• The Lenard scheme produces an infinite sequence of closed 1-forms $F_0, F_1, F_2, \ldots \in C^1(A, \mathcal{V})$.

• $\mathcal{V}$ is called normal if $\frac{\delta}{\delta u_i^{(n)}} : \mathcal{V} \to \mathcal{V}$ is surjective on each degree.

• $\mathcal{V}$ normal $\Rightarrow C^\bullet(A, \mathcal{V})$ is exact.

• Infinite sequence of integrals of motion: $\int h_n = (\frac{\delta}{\delta u})^{-1} F_n$.

• Integrable hierarchy: $\frac{du}{dt_n} = \{\int h_n, u\}_0$. 