Calculus Structure on the Variational Complex

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54th Annual Meeting of the AustMS
The University of Queensland
27-30 September 2010

Outline

1. Lie conformal algebras
   - Motivation from QFT
   - Definition

2. Cochain and chain complexes
   - Definitions
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3. Variational complex
   - Construction using Lie conformal algebras
   - Relevance to integrable hierarchies

Quantum fields in 2 dimensions

- Quantum fields are operator-valued distributions.

- Basic problem: For $x, y \in \mathbb{R}^d$,

$$\lim_{x \to y} \phi(x)\psi(y) = ?$$

- When $d = 2$, introduce

$$z = x^0 - x^1, \quad \bar{z} = x^0 + x^1 \quad \Rightarrow \quad |x|^2 = z\bar{z}$$

- Quantum fields are elements in $\mathrm{End}(\mathcal{H})[[z, z^{-1}, \bar{z}, \bar{z}^{-1}]]$.

- Locality axiom:

$$(z - w)(\bar{z} - \bar{w}) < 0 \quad \Rightarrow \quad [\phi(z, \bar{z}), \psi(w, \bar{w})] = 0$$

Operator product expansion

- Chiral quantum fields: $\partial_{\bar{z}}\phi(z,\bar{z}) = 0.$
- Locality axiom for chiral quantum fields:

$$(z-w)^N[\phi(z),\psi(w)] = 0, \quad N >> 0$$

- Operator product expansion:

$$\phi(z)\psi(w) = \sum_{j=0}^{N-1} \frac{\phi(w)_{(j)}\psi(w)}{(z-w)^{j+1}} + :\phi(z)\psi(w):$$

where

$$\phi(w)_{(j)}\psi(w) = \mathrm{Res}_z (z-w)^j [\phi(z),\psi(w)]$$

λ-bracket

- The j^{th}-products satisfy the Borcherds identity:

$$\sum_{j\in\mathbb{Z}_+}(-1)^j\binom{n}{j}\Big(\phi_{(m+n-j)}(\psi_{(k+j)}\chi)-(-1)^n\psi_{(k+n-j)}(\phi_{(m+j)}\chi)\Big)$$

$$=\sum_{j\in\mathbb{Z}_+}\binom{m}{j}(\phi_{(n+j)}\psi)_{(m+k-j)}\chi$$

for all $m, n, k \in \mathbb{Z}$ and quantum fields ϕ, ψ and χ.

- Generating function:

$$[\phi_\lambda\psi]=\sum_{j\in\mathbb{Z}_+}(\phi_{(j)}\psi)\frac{\lambda^j}{j!}$$

Lie conformal algebras

- A Lie conformal algebra is a $\mathbb{C}[\partial]$-module A with a λ-bracket $[\cdot_\lambda \cdot] : A \otimes A \to \mathbb{C}[\lambda] \otimes A$, satisfying
  - *Sesquilinearity*
$$[\partial a_\lambda b] = -\lambda [a_\lambda b]$$
$$[a_\lambda \partial b] = (\partial + \lambda)[a_\lambda b]$$
  - *Skewsymmetry*
$$[a_\lambda b] = -[b_{-\partial-\lambda} a]$$
  - *Jacobi identity*
$$[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [b_\mu [a_\lambda c]]$$

- Note: $[\cdot_\lambda \cdot]|_{\lambda=0}$ defines a Lie bracket on $A/\partial A$.

Modules

- A module over a Lie conformal algebra A is a $\mathbb{C}[\partial]$-module M endowed with a λ-action $A \otimes M \to \mathbb{C}[\lambda] \otimes M$, denoted $a \otimes m \mapsto a_\lambda m$, such that
  - $(\partial a)_\lambda m = -\lambda a_\lambda m$
  - $a_\lambda(\partial m) = (\partial + \lambda) a_\lambda m$
  - $a_\lambda(b_\mu m) - b_\mu(a_\lambda m) = [a_\lambda b]_{\lambda+\mu} m$

Example: Virasoro conformal algebra

- Centerless Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n}$$

- Quantum fields: $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

$$L(z)L(w) = \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)} + :L(z)L(w):$$

- Associated Lie conformal algebra:

$$Vir = \mathbb{C}[\partial]L\,, \qquad [L_\lambda L] = (\partial + 2\lambda)L$$

- All finite irreducible non-trivial Vir-modules are

$$M_{\Delta,\alpha} = \mathbb{C}[\partial]v\,, \qquad L_\lambda v = (\partial + \alpha + \Delta\lambda)v$$

where $\Delta \in \mathbb{C}^*, \alpha \in \mathbb{C}$. (Cheng-Kac, 1997)

Lie conformal algebra complex

- No Weyl complete reducibility theorem or Levi decomposition.
- (Bakalov-Kac-Voronov, 1999)

$$\Gamma^\bullet(A, M) = \tilde{\Gamma}^\bullet(A, M)/\partial\tilde{\Gamma}^\bullet(A, M)$$

- Drawbacks:
  - torsion information lost,
  - too small when $A \neq \mathrm{Tor}(A) \oplus (\mathbb{C}[\partial] \otimes U)$.

Cochain complex (De Sole-H-Kac, 2010)

- Let $\mathbb{C}_-[\lambda_1, \ldots, \lambda_k]$ denote the space of polynomials with the $\mathbb{C}[\partial]$-module structure:

$$\partial P(\lambda_1, \ldots, \lambda_k) = -(\lambda_1 + \cdots + \lambda_k) P(\lambda_1, \ldots, \lambda_k)$$

- The space of *k*-cochains $C^k(A, M)$ is the space of $\mathbb{C}$-linear maps

$$\begin{aligned} c \quad : \quad A^{\otimes k} &\longrightarrow \mathbb{C}_-[\lambda_1, \ldots, \lambda_k] \otimes_{\mathbb{C}[\partial]} M \\ a_1 \otimes \cdots \otimes a_k &\mapsto c_{\lambda_1, \cdots, \lambda_k}(a_1, \cdots, a_k) \end{aligned}$$

satisfying certain *skewsymmetry* and *sesquilinearity* relations.

Example: Virasoro conformal algebra cohomology

- For finite Lie conformal algebras A:

$$C^\bullet(A, M) \cong \frac{\Delta^\bullet(\mathrm{Lie}_- A, M)}{\partial \Delta^\bullet(\mathrm{Lie}_- A, M)}$$

- $\mathrm{Lie}_- \mathit{Vir} = \mathrm{Vect}(\mathbb{C})$.

- Non-trivial cohomology only when $\Delta = 1 - \frac{(3r^2 \pm r)}{2}$ and $\alpha = 0$:

$$\dim H^q(\mathit{Vir}, M_{\Delta,\alpha}) = \begin{cases} 2 & \text{for } q = r + 1 \\ 1 & \text{for } q = r \text{ or } r + 2 \end{cases}$$

Chain space

- Let $\mathcal{M}_k \subset M[[x_1, \dots, x_k]]$ denote the subspace of formal power series, such that

$$\partial\phi(x_1, \dots, x_k) = (\partial_{x_1} + \cdots + \partial_{x_k})\phi(x_1, \dots, x_k)$$

- The space of *k-chains* $C_k(A, M)$ is the quotient of $A^{\otimes k} \otimes \mathcal{M}_k$ by the relations
  - $a_1 \otimes \dots \partial a_i \cdots \otimes a_k \otimes \phi = -a_1 \otimes \cdots \otimes a_k \otimes (\partial_{x_i}\phi)$
  - $\sigma(a_1 \otimes \cdots \otimes a_k \otimes \phi) = \mathrm{sign}(\sigma) a_1 \otimes \cdots \otimes a_k \otimes \phi, \quad \sigma \in S_k$

- $C_\bullet(A, M)$ is not a complex in general.

Calculus structure

- $C_\bullet(A, M)$ is a Gerstenhaber algebra.
- Bracket on 1-chains:
  1. The pair (A, M) is a Lie conformal algebroid.
  2. Extends to a Lie conformal algebroid $(M \otimes A, M)$ with λ-bracket
  $$[(m \otimes a)_\lambda (n \otimes b)]' = ((e^{\partial^M \partial_\lambda} m)n) \otimes [a_\lambda b]$$
  3. Bracket on $C_1(A, M) \cong (A \otimes M)/\partial(A \otimes M)$ is $[\cdot_\lambda \cdot]'|_{\lambda=0}$.
- *Contraction operator* and *Lie derivative* provide a representation of $C_\bullet$ on the complex $(C^\bullet, d)$.

Calculus of variations

- Basic problem: Minimize

$$S = \int_a^b f\left(u_i, u_i', u_i'', \ldots, u_i^{(n)}\right) dx, \qquad u_i \in \mathcal{V},\ i \in I = \{1, \ldots, N\}$$

- Euler-Lagrange equations:

$$\frac{\delta f}{\delta u_i} := \sum_{k=0}^{n} (-\partial)^k \frac{\partial f}{\partial u_i^{(k)}} = 0, \quad i \in I$$

- Variational derivative: $\frac{\delta}{\delta u} : \mathcal{V} \to \mathcal{V}^N$.

- Basic identity:

$$\left[\frac{\partial}{\partial u_i^{(n)}}, \partial\right] = \frac{\partial}{\partial u_i^{(n-1)}} \quad \Rightarrow \quad \partial\mathcal{V} \subset \operatorname{Ker}\frac{\delta}{\delta u}$$

Variational complex

- (Helmholtz) $F \in \operatorname{Im} \frac{\delta}{\delta u} \quad \Rightarrow \quad D_F(\partial) = D_F(\partial)^*$ where

$$D_F(\partial)_{ij} = \sum_{n \in \mathbb{Z}_+} \frac{\partial F_i}{\partial u_j^{(n)}} \partial^n$$

- Restatement:

$$\mathcal{V}/\partial\mathcal{V} \xrightarrow{\frac{\delta}{\delta u}} \mathcal{V}^N \xrightarrow{D_\bullet(\partial) - D_\bullet(\partial)^*} \Omega^2(\mathcal{V}) \to \dots$$

- Question: what are the higher cochains and differentials?

Construction of the variational complex

- Define $A = \oplus_{i \in I} \mathbb{C}[\partial] u_i$ with trivial λ-bracket.
- Define a representation of A on $\mathcal{V}$ by

$$u_{i\lambda} f = \sum_{n \in \mathbb{Z}_+} \lambda^n \frac{\partial f}{\partial u_i^{(n)}}, \quad i \in I$$

  and extend by sesquilinearity.
- The variational complex is $(C^\bullet(A, \mathcal{V}), d)$.

Higher cochains and differentials

- For $k \geq 2$, elements in $C^k(A, \mathcal{V})$ are *skewsymmetric k-differential operators* $S : (\mathcal{V}^N)^k \to \mathcal{V}/\partial\mathcal{V}$ of the form

$$S(P^1, \dots, P^k) = \int \sum_{\substack{i_r \in I \\ n_r \in \mathbb{Z}_+}} f^{n_1, \dots, n_k}_{i_1, \dots, i_k} (\partial^{n_1} P^1_{i_1}) \cdots (\partial^{n_k} P^k_{i_k})$$

- The differential is

$$(dS)(P^1, \dots, P^{k+1}) = \sum_{s=1}^{k+1} (-1)^{s+1} (X_{P^s} S)(P^1, \overset{s}{\check{\cdots}}, P^{k+1})$$

- $C_\bullet(A, \mathcal{V})$ is the Gerstenhaber algebra of *evolutionary* polyvector fields.

Construction of integrable hierarchies

- Bi-hamiltonian system: $\frac{du}{dt} = \{\int h_1, u\}_0 = \{\int h_0, u\}_1$.
- The Lenard scheme produces an infinite sequence of closed 1-forms $F_0, F_1, F_2, \dots \in C^1(A, \mathcal{V})$.
- $\mathcal{V}$ is called normal if $\frac{\delta}{\delta u_i^{(n)}} : \mathcal{V} \to \mathcal{V}$ is surjective on each degree.
- $\mathcal{V}$ normal $\Rightarrow C^\bullet(A, \mathcal{V})$ is exact.
- Infinite sequence of *integrals of motion*: $\int h_n = \left(\frac{\delta}{\delta u}\right)^{-1} F_n$.
- Integrable hierarchy: $\frac{du}{dt_n} = \{\int h_n, u\}_0$.