Some Thoughts on Physics

Auckland, New Zealand Jan 10, 2020

Michael Freedman, Microsoft Research

I. Can many body physics emerge from single particle quantum mechanics via symmetry breaking?

Phrased dramatically: Could the **universe** really just be a **single particle** in a “flat band”?

II. Is there a duality between curvature and topology? We are used to Maxwell terms $|F \wedge {}^{*}F|$ and metric curvature R showing up in Lagrangian densities. Can these terms be replaced with **topology** and **distortion** densities?

III. Field Theory (FT) has proven difficult for mathematics to swallow. Should we stop trying and instead use FT as a fundamental concept for mathematics? Could path integrals establish a *truth value* in $\mathbb{C} \cup \infty$ based on preponderance of evidence?

We introduce Topic I by considering the Riemannian Geometry of left invariant (l.i.) metrics on a Lie Group.

Math: Berger 1960, Milnor 1976, Boucette 2016

Physics: Nielsen et al. ~2005-2008 and Brown/Susskind 2017, l.i. metric encodes *allowed* or *likely* interactions.

Consider n-qubits $= (\mathbb{C}^2)^{\otimes n} \cong \mathbb{C}^{2^n}$ with symmetry group $\mathrm{SU}(2^n)$.

$$\mathrm{su}(2^n) = \mathrm{span}\{\text{Pauli words}\} = \{\underbrace{I \otimes X \otimes Y \otimes I \otimes I \otimes Z \otimes Z \otimes I}_{n}\}$$

where $I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $Z = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Consider l.i. metric $g_{ij}^b = b^{w(i)}\delta_{ii}$, where i, j index Pauli words, $w(i) = \#\text{ of } X, Y, \text{or } Z$,

$b > 0 = \text{base}, b \gg 1.$

Describes a geodesic distance $\sim$ compilation length of a quantum algorithm

This resembles a bosonic version of SYK dynamics: $H = \sum J_{ijkl}\gamma_i\gamma_j\gamma_k\gamma_l$

SYK is a natural fermionic alternative to the Nielsen/Brown/Susskind metric.

$\mathrm{SU}(2^n) = \mathrm{span}\,\{\gamma\text{-words}\} = \{\overbrace{(l \otimes \gamma \otimes \gamma \otimes l \otimes 1 \otimes l \otimes \gamma \otimes l \otimes \gamma)}^{2n}\}$ and $g_{ij} = b'^{\,w'(i)}\delta_{i,i}$, where i, j index γ-words,

$w'(i) = \#\text{ of }\gamma,$ $b' > 1 = \text{base}.$

This metric is important to understanding time scales in black holes: thermalization, scrambling, etc.

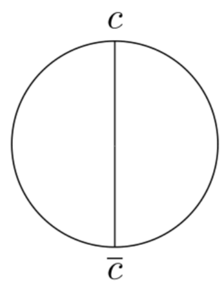
Can either of these metrics, which “**know about qubits**” or “**square root of qubits**”, arise via symmetry breaking ?

The idea is that the difference between single particle physics and many body physics is only in a matter of what’s “**typical**” for the system **Hamiltonian.**

And what’s typical is a matter of the metric on the Lie algebra su(Hilbert space) of system symmetries.

So let’s consider some simple functionals that might do the job.

Consider the simplest scalar, θ, associated with a l.i.m. on a real Lie group G.

$\theta = c_{ij}^{k'} g_{k'k} \bar{c}^{k}_{i'j'} g^{i'i} g^{j'j} =$ 

, built from structure constants and metric,

where $[e_i, e_j] = c_{ij}^{k} e_k$.

We should normalize $\bar{\theta} = \frac{\theta}{\det(g)^{2/\dim}}$.

Unfortunately, θ achieves a unique minimum at the bi-invariant metric – it does not break symmetry

But consider $\boldsymbol{f}$ defined to be $\int (\|[a,b]\|^2 \; \|[b,c]\|^2 \; \|[c,a]\|^2)$ over g-random triples of Hamiltonians. **Modji Shokrian-Zini** and I have been numerically studying such functionals.

$\boldsymbol{f}$ is sensitive to the **quasi-clique** structure of the "commutivity graph" possessed by interacting metrics:

For $w(i), \mathrm{w}(\mathrm{j}) \ll \sqrt{n}$ the pigeonhole principle says Pauli-word-basis elements e_i and e_j almost surely commute. so by minimizing $\boldsymbol{f}$ we hope $\mathrm{SU}(2^n)$ will find something akin to the N-B-S geometry.

On the Hilbert space level this would mean minimize $\boldsymbol{f}$ finds qubits inside $\mathbb{C}^{2^n}$:

After *symmetry breaking via* $\boldsymbol{f}$, the typical Hamiltonian should be organized as a typical many-body Hamiltonian with only a few bodies per interaction.

Do you remember from ***The Hitchhiker's Guide to the Galaxy*** the answer the super (quantum?) computer gave to the "meaning of life"?
It was "42."

That is, $2 \cdot 3 \cdot 7.$

Now in our universe, we have at least a Googol (10^{100}) degrees of freedom, so our Hilbert space has at least Googolplex dimension:

$$\mathbb{C}^N, \qquad N \approx 2^{10^{100}}$$

If the early universe attempted to minimize $\boldsymbol{f}$ on the special unitary group $(\mathbb{C}^N)$, and this is the origin of the apparent many-body physics we see around us, then

The statistics of the factorization of a large random number N might leave some observable signature at low energy.

What might it be? (Compare Golomb-Dickman Theorem).

II. Faking Curvature

Milnor-Wood phenomenon:

Suppose
$$\begin{array}{ccc} S^1 & \longrightarrow & B \\ & & \downarrow \\ & & \Sigma^2 \end{array}$$
is a circle bundle over a closed surface Σ^2. Then B admits a flat PSL$(2, R)$-connection iff $|\chi(B)| \leq |\chi(\Sigma)|$.

This is a Surprise: Because for $\mathrm{U}(1)$-connection A, $\chi_B = \int_\Sigma \epsilon_{i_1,\ldots,i_n} \Omega_{i_1,i_2} \ldots \Omega_{i_{n-1},i_n}$.

Including $\mathrm{U}(1) \subset \mathrm{PSL}(2, R)$ kills the integral formula, and

Principle bundle → associated bundle.

Fact: If $0 < \log|\mathrm{tr}(A) - 2| < \epsilon$, then we need $|\chi(\Sigma)| \approx \frac{2\pi}{\epsilon^2}\chi(B)$

The genus of Σ must grow if the connection is required to be nearly unitary.

But if the genus of Σ is large enough, deviation from unitarity can be *imperceptible*: topology can *fake* curvature.

Curiously, $\chi^2 = p_1$ is more stable. For any $\mathrm{GL}(n)$-connection

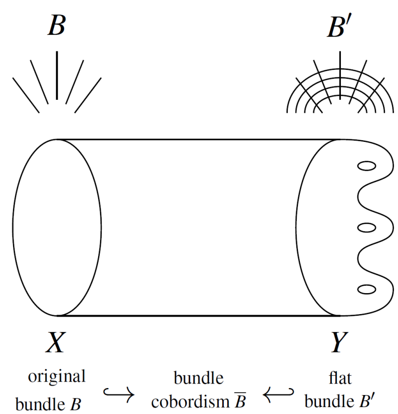
$$\sum \delta^{j_1 \dots j_{2k}}_{i_1 \dots i_{2k}} \Omega^{i_1}_{j_1} \cdots \Omega^{i_{2k}}_{j_{2k}}$$

integrates to a multiple of p_1.

Mather-Thurston theorem

Mather-Thurston theorem (1974): Given any manifold M (noncompact OK), $B\mathrm{Homeo}(M)^\delta \xrightarrow{\mathrm{id}_*} B\mathrm{Homeo}(M)$ is a homology isomorphism.

This means that every bundle with fiber M is bordant to a bundle with a foliation transverse to the fibers (i.e. to one with a flat connection).



Also proved for $\mathbb{C}^1$(Tsuboi Annals 1989), false for $\mathbb{C}^2$ (Bott).

Conjectural extensions of Mather-Thurston (thanks to Shmuel Weinberg, Chaitanya Murthy, and Sam Nariman for discussions)

A. The cobordism W from X to Y should admit a degree 1 map to the *original* end, $f: W \to X$

A'. Possibly $f: W \to X$ can be arranged to be a simple deformation retraction when $\dim(X) \geq 3$ (making W a "semi s-cobordism", in which case X is a "plus-construction" on Y.

B. If $\begin{array}{ccc} M & \longrightarrow & B \\ & & \downarrow \\ & & X \end{array}$ has structure group $K \subset \mathrm{Homeo}(M)$ (or $\mathrm{Diff}^1(M)$) then for any ϵ-neighborhood

$\mathcal{N}_\epsilon(K) \subset \mathrm{Homeo}(M)$ (or $\mathrm{Diff}^1(M)$) W can produced with the structure group the bundle

bordism in $\mathcal{N}_\epsilon(K)$.

C. A and B can be simultaneously achieved.

Fact: A) holds at least when dim of base $\dim(X) = 3$ i.e. X cobordant to Y with the bordism admitting a map $W \xrightarrow{\deg 1} X$.

Pf: Give X a Heegaard decomposition $X = HB_1 \cup_\Sigma HB_2$. B may be flattened over HB_1 since HB_1 is 1D.

Now use a relative version of Mather-Thurston over HB_2, and the fact that handlebodies are terminal objects in the category of bounded manifolds and degree one maps.

Conjecture D: Faking Curvature (principle bundle case, fiber G)

Rough statement: In addition to achieving C - "roughen" the topology and "roughen" the transitions instead of curvature), by introducing additional topology on a tiny scale ϵ_0, then at a slightly larger scale ϵ_1, the flat connection on B' can be engineered to approximate, through its holonomy, the original curvature operator on X,

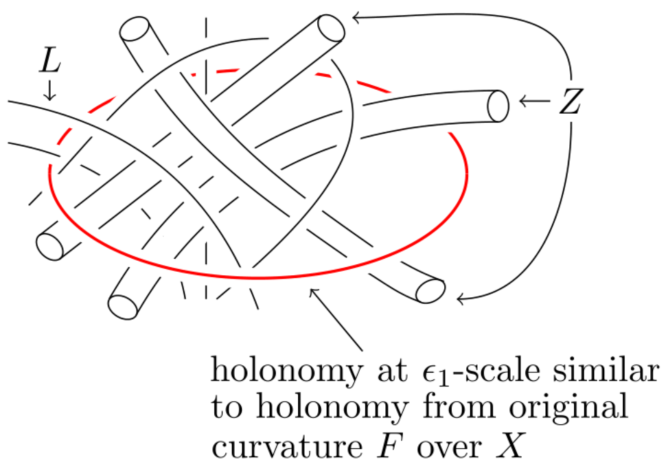
$F = DA + A \vee A$.

I believe I can prove this when $\dim(X) \leq 3$.

The idea is to fill Y with a *very* dense link or knot L statistically uniform in its sampling of $T_1(Y)$, the unit tangent bundle of Y. Substituting $\mathcal{N}(L)$ with a topologically richer Z define:

$$Y' = \big(Y - \mathcal{N}(L)\big) \cup Z$$

Playing with the representation $\pi_1(Z) \to \mathrm{Homeo}(G)$ allows one to create holonomies at scale ϵ_1 very similar to the original curvature F over X.



With some modification, the principle bundle story should adapt to the tangent bundle, allowing gravity also to be simulated by a flat connection over Y, a "roughened version of X" , again with the transition functions only **slightly** outside the orthogonal group.

There is a diagram, with the upper arrow also classifying the tangent bundle:

$$\begin{tikzcd}
Y \arrow[rr, dashed, "\tau'"] \arrow[d, hook] & & B\mathrm{Top}(n)^{\delta} \arrow[dd] \\
W \arrow[drr] & & \\
X^n \arrow[u, hook] \arrow[r, "\tau"] & BO(n) \arrow[r] & B\mathrm{Top}(n)
\end{tikzcd}$$

Can the Riemannian metric itself emerge from topology?

If space X is stringy (and, initially, 3D), say $X = X_0 \times S^1$, then replacement yields

$$Y = \left(X \setminus \mathcal{N}(L)\right) \cup Z$$

And then the discrete metric:

$$\mathrm{dist}_{\mathrm{X}_0}(x, x') \coloneqq \mathrm{dist}_{\mathrm{X}}(x \times S^1, x' \times S^1) \coloneqq \min_{\partial\Sigma = x \times S^1 \coprod -x' \times S^1} \mathrm{bordisms}\ \Sigma \subset Y \quad [\mathrm{genus}\ \Sigma]$$

can be rescaled to approximate the original Riemannian metric on X.

What would be the upshot if this all works out?

The goal is a ***duality*** for Lagrangian densities which allows the Maxwell term ($F \wedge {}^*F$) and the Einstein-Hilbert action R to be replaced with dynamical fields:

1. Topology density ρ, and a measure of deviation from a compact Lie group of symmetries
2. $\mathrm{dev}(A)$

One can summarize the goal as:

$$\text{Mass} = \text{Energy} = \text{Topology}$$

The idea is that living in the IR, one cannot tell the difference between curvature and a flat connection over a base space whose fundamental group has been locally enhanced—perhaps merely through an extension by a perfect group.

The dynamics of this dual theory would create and destroy microscopic topology at will.

It seems natural to expect that the ***topology field*** be a **quantum field**, so that no single enhancement Y of X is picked out, but rather Y is fluctuating as well as dynamical. There should also be a **deviation field**, dev.

To obtain a duality with gauge theories $\mathcal{L}$ should contain Maxwell terms like $\|\rho\|^2$ and $\|\mathrm{dev}\|^2$, and possibly additional kinetic terms involving derivatives of ρ and dev, and conjugate terms :: $\rho\mathrm{dev}$

III. Foundations

The single most important idea in quantum field theory is that ***everything*** (“not impossible” in Feynman's words) is taken into account and contributes to the answer.

The 70-year-old math puzzle is that ***everything*** is rather **big** and hard to define integrals over.

Physics seems to operate closer to the legal standard **“preponderance of evidence”** than a platonic mathematical idealization.

Instead of trying to force physics into mathematics, maybe we should learn something new and design mathematical foundations which are *physics-inspired*.

Why might this be a good idea?

Wigner's phrase: "unreasonable effectiveness of mathematics in physics" no longer holds much surprise. Math, by now, seems obviously useful to keeping track of calculations and organizing concepts.

What is more interesting is ***converse Wigner***: Physics is unreasonably effective in pointing mathematics in the right direction.

If this assertion needs any argument, it is just two words: **Ed Witten**.

Remarkably among all the possible situations, conditions, equations, and objects that a mathematician might choose to study, physics has proven a reliable guide.

So perhaps we should trust it to form a new logical foundation for mathematics.

I'm envisioning fragments of arguments, like paths in a function space could be integrated to produce a truth value in $\mathbb{C} \cup \infty$, 0 corresponding to false and ∞ corresponding to true.

At the moment, this is merely a dream, but a pleasant one.

I don’t really know how to set up “path integrals over argument fragments,” or what **constructive** and **destructive interference** would be telling us.

Would “classical arguments” be stationary trajectories and the **Hessian** a measure ***confidence***?

Complexity theory might be a test case to keep in mind. Uniquely it is a branch of mathematics which floats in the air. Nothing absolute is known; it is all about relations and implications.

A final thought on this topic.

Everything we know about logic seems to descend from Gödel. But if proofs become path integrations, perhaps they can no longer be enumerated and enter other proofs as variables, so self-reference might not be expressible. **I wonder if a physics-based foundation for mathematical logic might be free of undecidable statements.**