Knot Categorification
from Geometry and Mirror Symmetry

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I will describe

two geometric approaches to the

knot categorification problem,

which emerge from string theory.

The relation between our

two approaches

is a variant of

two dimensional mirror symmetry.

Mirror symmetry turns out to play a crucial role

in essentially all aspects of the problem.

One of approaches is in the same spirit as that of

Kamnitzer and Cautis.

The other approach uses symplectic geometry.

It is a cousin of the approach of

Seidel and Smith,

who pioneered such geometric approaches to problem,

but produced a theory not sufficiently rich.

There is a third approach
with the same string theory origin,
due to Witten.

What emerges from string theory
is unified framework
for knot categorification.

The story I will tell you about originated from joint work

with Andrei Okounkov,

where the main focus was integrable lattice models,

rather than knot invariants.

Here, I will focus on what is directly relevant to knot theory.

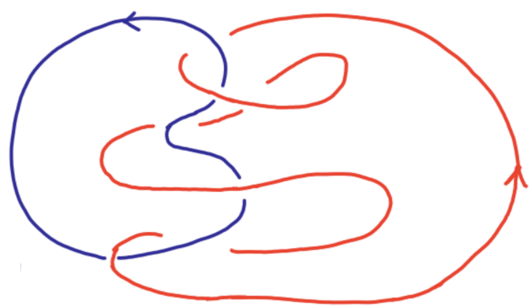
Some mirror symmetry aspects of the story are also new,

and will appear in joint work with Vivek Shende and Michael McBreen.

To begin with, it is useful to recall

some well known aspects of knot invariants.

A quantum invariant of a link



depends on a choice of a Lie algebra,

$$^{L}\mathfrak{g}$$

and a coloring
of its strands by representations of $^{L}\mathfrak{g}$.

The link invariant,

in addition to the choice of a group

$$^L\mathfrak{g}$$

and its representations,

depends on one parameter

$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

Edward Witten explained in ’89 that,

if κ is integer, the knot invariant

comes from

Chern-Simons theory with gauge group based on the Lie algebra

$$^{L}\mathfrak{g}$$

and (effective) Chern-Simons level

$$\kappa$$

In the same paper he also showed that
underlying Chern-Simons theory is a
two-dimensional conformal field theory associated to

$$\widehat{{}^L\mathfrak{g}}_\kappa$$

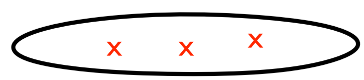
an affine Lie algebra of ${}^L\mathfrak{g}$, at level κ.

We will take this as the starting point.

Start with the space conformal blocks of

$$\widehat{{}^{L}\mathfrak{g}}_{\kappa}$$

on a Riemann surface $\mathcal{A}$ with punctures

$\mathcal{A}$ 

To eventually get invariants of knots in $\mathbb{R}^3$ or S^3

we want to take

$$\mathcal{A}$$

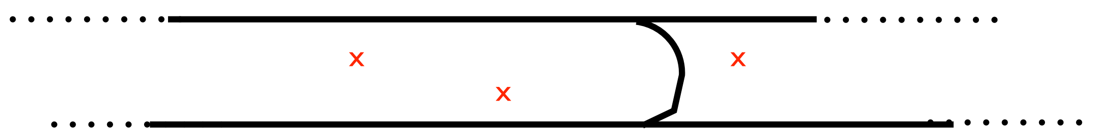
to be a complex plane with punctures.

$\mathcal{A}$ 

It is equivalent, but better for our purpose,

to take it to be a punctured infinite cylinder.

Conformal blocks of $\widehat{{}^L\mathfrak{g}_\kappa}$ on $\mathcal{A}$



are the following correlators of chiral vertex operators

$$\Psi(a_1, \ldots, a_n) = \langle\lambda|\, \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n)\, |\lambda'\rangle$$

A chiral vertex operator

$$\Phi_{{}^L\rho_I}(a_I)$$

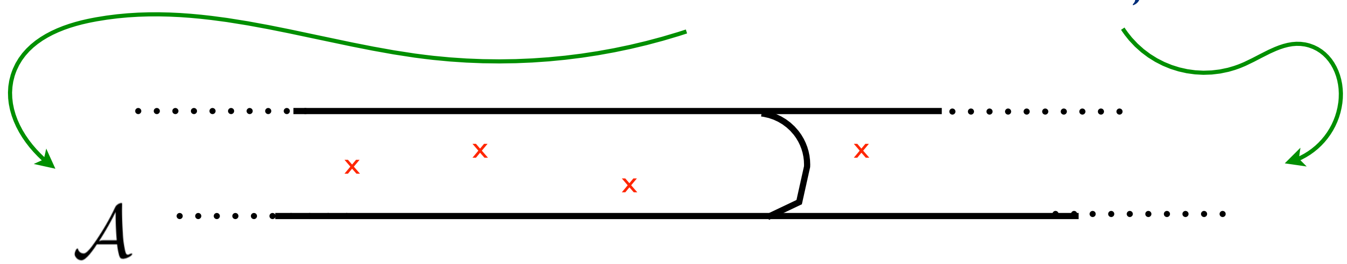
associated to a finite dimensional representation

$${}^L\rho_I$$

of ${}^L\mathfrak{g}$ adds a puncture at a finite point on $\mathcal{A}$

$$x = a_I$$

Punctures at $x = 0$ and ∞ ,



are labeled by a pair of highest weight vectors

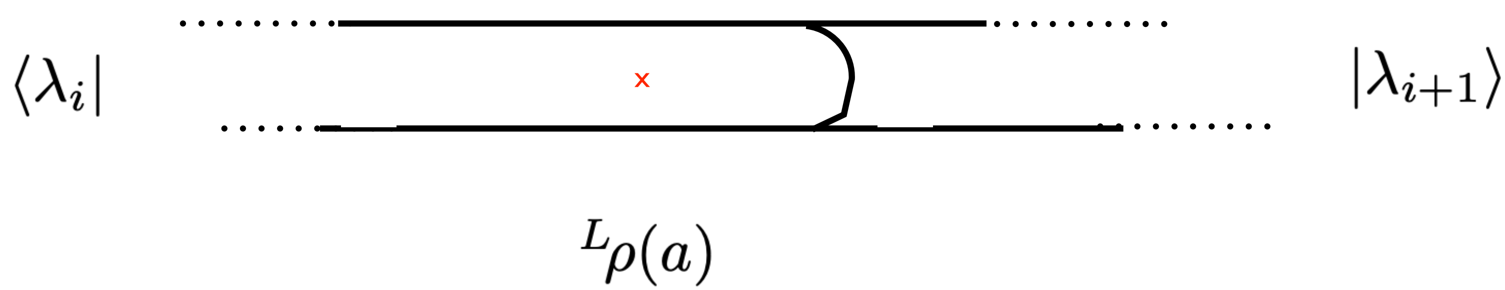
$|\lambda\rangle$ and $|\lambda'\rangle$

of Verma module representations of the algebra.

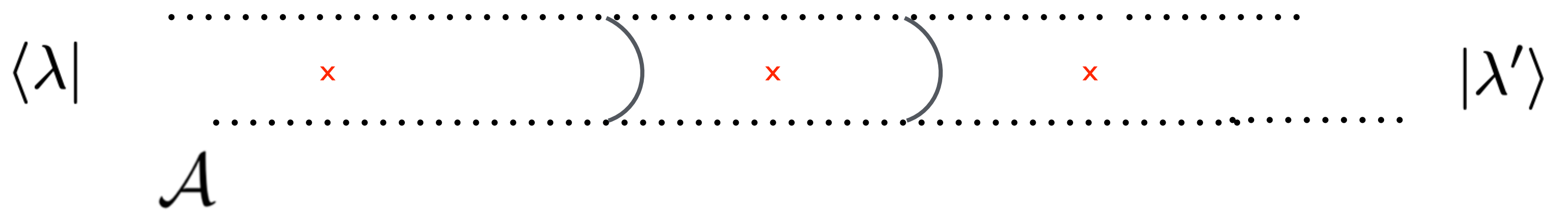
A chiral vertex operator acts as an intertwiner

$$\Phi_{{}^L\rho}(a) : \rho_{\lambda_i,k} \to \rho_{\lambda_{i+1},k} \otimes {}^L\rho(a)$$

between a pair of Verma module representations,



Sewing the chiral vertex operators



we get different conformal blocks,

$$\Psi(a_1, \ldots, a_n) = \langle\lambda|\ \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n)\ |\lambda'\rangle$$

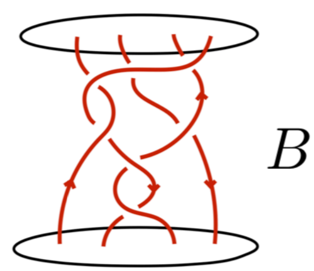
depending on choices of intermediate Verma module representations.

(which the notation hides).

The Chern-Simons path integral on

$$\mathcal{A} \times \text{interval}$$

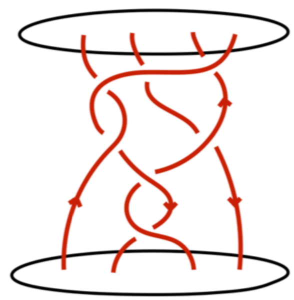
in the presence of a braid



gives the corresponding
quantum braid invariant.

The braid invariant

$$\mathfrak{B} = \mathfrak{B}(B)$$



is a matrix that transports

the space of conformal blocks,

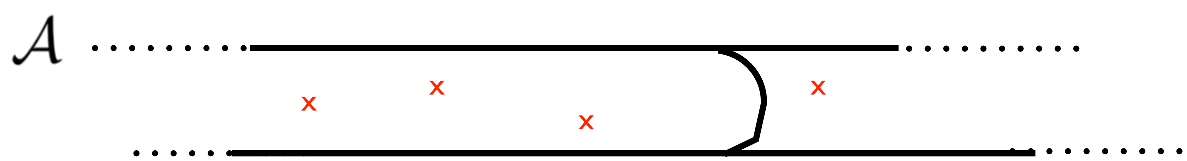
along the braid B

To describe the transport,

instead of characterizing $\widehat{{}^L\mathfrak{g}_\kappa}$ conformal blocks

$$\Psi(a_1, \ldots, a_n) = \langle \lambda | \, \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \, | \lambda' \rangle$$

in terms of vertex operators and sewing,



it is better to describe them as solutions to a differential equation.

The equation solved by conformal blocks of $\widehat{{}^L\mathfrak{g}}_\kappa$ on $\mathcal{A}$

$$\Psi(a_1,\ldots,a_n) = \langle\lambda|\, \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\,|\lambda'\rangle$$

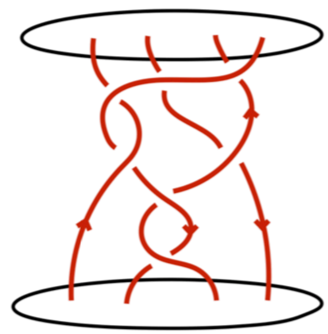
is the equation discovered by Knizhnik and Zamolodchikov in '84:

$$\kappa\, a_I \frac{\partial}{\partial a_I}\Psi = \left(\sum_{J\neq I} r_{IJ}(a_I/a_J) + r_{I0} + r_{I\infty}\right)\Psi$$

The equation makes sense for any $\kappa \in \mathbb{C}$, not necessarily integer.

The quantum braid invariant

$$\mathfrak{B}(B)$$



is the monodromy matrix of the Knizhnik-Zamolodchikov equation,

along the path in the parameter space corresponding to

the braid B .

The monodromy problem of the $\widehat{{}^L\mathfrak{g}}_\kappa$ Knizhnik-Zamolodchikov equation

$$\kappa\, a_I \frac{\partial}{\partial a_I}\,\Psi = \sum_{J\neq I} r_{IJ}(a_I/a_J)\,\Psi$$

was solved by Tsuchia and Kanie in '88 and, Drinfeld and Kohno in '89.

They showed that its monodromy matrices are given in terms of the R-matrices of the quantum group

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to ${}^L\mathfrak{g}$

Action by monodromies

turns the space of conformal blocks into a module for the

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

quantum group in representation,

$${}^L\rho = \otimes_I {}^L\rho_I$$

The representation ${}^L\rho$ is viewed here as a representation of $U_{\mathfrak{q}}({}^L\mathfrak{g})$ and not of ${}^L\mathfrak{g}$, but we will denote by the same letter.

The monodromy action

is irreducible only in the subspace of

$$\Psi(a_1,\ldots,a_\ell,\ldots,a_n) \in ({}^L\rho_1 \otimes \ldots {}^L\rho_\ell \otimes \ldots \otimes {}^L\rho_n)_\nu$$

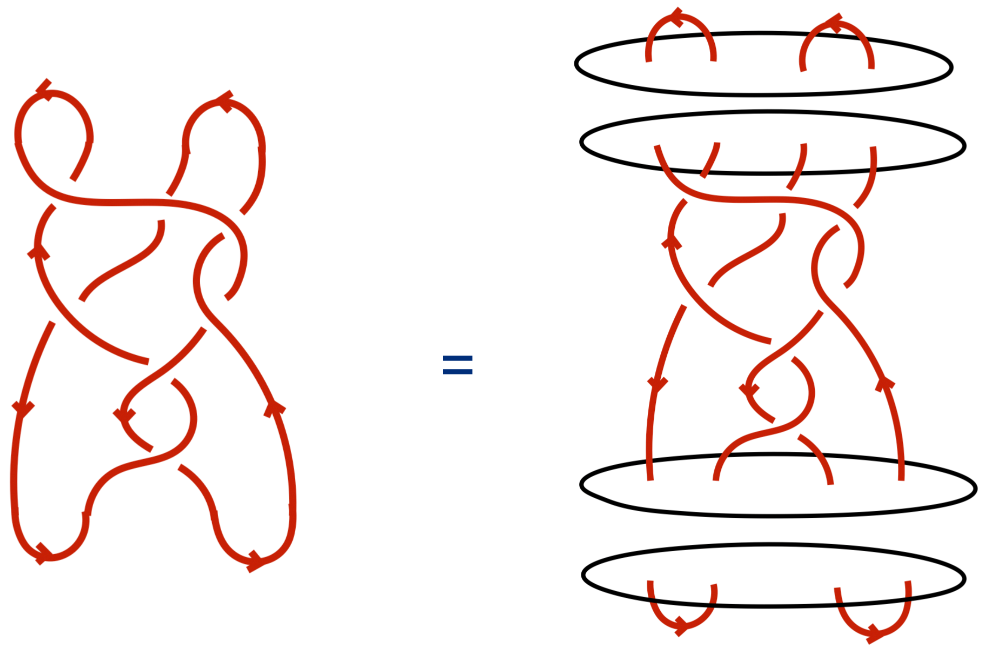
of fixed

weight $\nu = \lambda - \lambda'$

We will make use of that.

This perspective leads to
quantum invariants of not only braids
but knots and links as well.

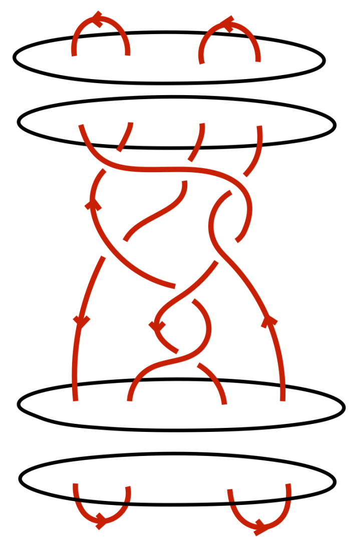
Any link K can be represented as a



a closure of some braid B

The corresponding quantum link invariant is the matrix element

$$\left(\Psi_{\mathcal{L}_{out}}\right| \mathfrak{B} \left|\Psi_{\mathcal{L}_{in}}\right)$$



$\mathfrak{B}(B)$

of the braiding matrix,

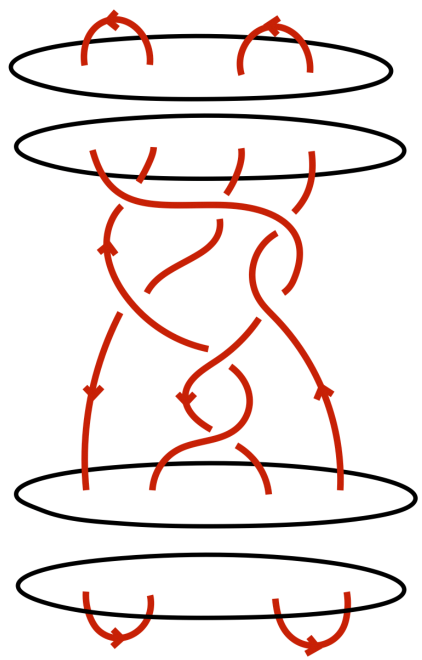
taken between a pair of conformal blocks

$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

The pair of conformal blocks

$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

that pick out the matrix element



$$\mathfrak{B}(B) = \langle \Psi_{out} | B | \Psi_{in} \rangle$$

correspond to the top and the bottom of the picture.

The conformal blocks

we need are specific solutions to KZ equations

$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

which describe pairwise fusing vertex operators

into copies of trivial representation.

Necessarily they correspond to subspace of

$$^L\rho = \otimes_I \, ^L\rho_I$$

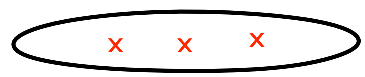
of weight

$$\nu = 0$$

To categorify quantum knot invariants,

one would like to associate

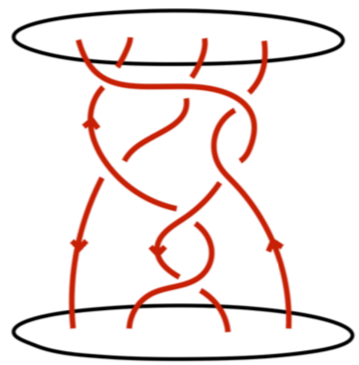
to the space conformal blocks one obtains at a fixed time slice

$\mathcal{A}$ 

a bi-graded category,

and to each conformal block an object of the category.

To braids,



one would like to associate

functors between the categories

corresponding to the

top and the bottom.

Moreover,
we would like to do that in the way that
recovers the quantum knot invariants upon
de-categorification.

One typically proceeds by coming up with a category,
and then one has to work to prove
that de-categorification gives
the quantum knot invariants one aimed to categorify.

The virtue of the two of the approaches
I am about to describe,
is that the second step is automatic.

I will start by describing the two
approaches we end up with,
and the relation between them
in a manner that is more or less self contained.

Later on,

I will describe their superstring theory origin

which is the same as in Witten's approach.

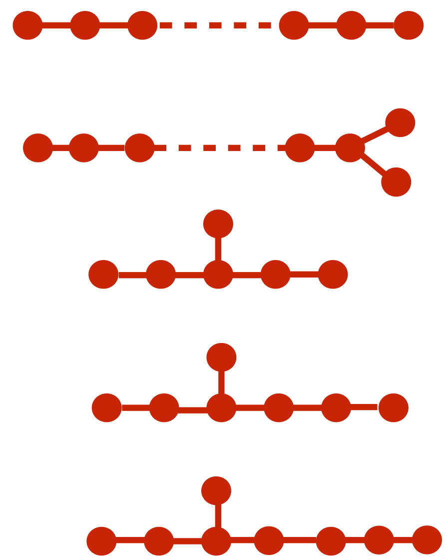
The starting point for us is

a geometric realization

Knizhnik-Zamolodchikov equation.

For the rest of the talk, we will specialize ${}^L\mathfrak{g}$ be a simply laced Lie algebra

so ${}^L\mathfrak{g} = \mathfrak{g}$ are one of the following types:



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

The generalization to non-simply laced Lie algebras

involves an extra step, we won't have time for.

It turns out that Knizhnik-Zamolodchikov equation of

$$\widehat{L\mathfrak{g}}$$

is the "quantum differential equation" of a certain holomorphic symplectic manifold.

This result has been proven very recently, by Andrei's student Ivan Danilenko.

Quantum connection of a Kahler manifold $\mathcal{X}$ is a connection

$$\nabla_D = a_D \frac{\partial}{\partial a_D} - D\star$$

on a (flat) vector bundle, with fiber $H^*(\mathcal{X})$.,

over the complex Kahler moduli.

The connection is defined in terms of quantum multiplication by divisors

$$D \in H^2(\mathcal{X})$$

Quantum multiplication used in the connection is a product on $H^*(\mathcal{X})$

$$\langle \alpha \star \beta, \gamma \rangle = \sum_{d \geq 0, d \in H^2(\mathcal{X})} (\alpha, \beta, \gamma)_d \; a^d$$

defined in terms of Gromov-Witten theory or,

the topological A-model of $\mathcal{X}$

Just as Knizhnik-Zamolodchikov equation

is central for many questions in representation theory,

quantum differential equation

is central for many questions in

algebraic geometry and in mirror symmetry.

We will be discovering here a new connection between the two.

To get the quantum differential equation

$$\nabla_D = a_D \frac{\partial}{\partial a_D} - D\star$$

to coincide with the Knizhnik-Zamolodchikov equation

$$\kappa a_\ell \frac{\partial}{\partial a_\ell} \Psi = \sum_{j \neq \ell} r_{\ell i}(a_\ell / a_j) \, \Psi$$

solved by conformal blocks of $\widehat{{}^L\mathfrak{g}}_k$,

$$\Psi(a_1, \ldots, a_n) = \langle \lambda | \, \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n) \, | \lambda' \rangle,$$

one wants to take $\mathcal{X}$ to be a very special manifold.

Recall that the conformal blocks take values

in the tensor product

$$\Psi(a_1, \ldots, a_\ell, \ldots, a_n) \in ({}^L\rho_1 \otimes \ldots {}^L\rho_\ell \otimes \ldots \otimes {}^L\rho_n)_\nu$$

of fixed weight

$$\text{weight } \nu = \lambda - \lambda'$$

The manifold

$$\mathcal{X}$$

we need can be described as the moduli space of

G monopoles on

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$$

where G is an adjoint form of a group with Lie algebra $\mathfrak{g}$

To a vertex operator

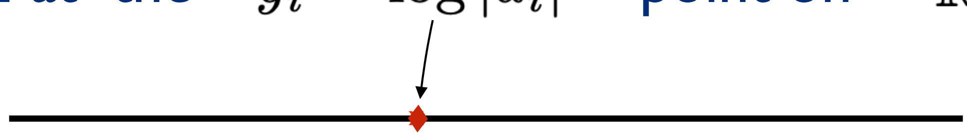
$$\Phi_{{}^L\rho_i}(a_i)$$

corresponds a singular G monopole of charge

$${}^L w_i$$

the highest weight of ${}^L\rho_i$,

located at the $y_i = \log|a_i|$ point on $\mathbb{R}$.



The choice of $\text{weight}\ \nu$ in:

$$({}^L\rho_1 \otimes \ldots {}^L\rho_\ell \otimes \ldots \otimes {}^L\rho_n)_\nu$$

determines the total monopole charge,

including that of smooth monopoles,

whose positions on $\mathbb{R} \times \mathbb{C}$ we get to vary

The complex dimension of the monopole moduli space is determined from the

$$\text{highest weight } \mu = \sum_a m_a \, {}^L w_a$$

and weight

$$\nu = \sum_a m_a \, {}^L w_a - \sum_a d_a \, {}^L e_a \,, \quad d_a \geq 0$$

as

$$\dim_{\mathbb{C}} \mathcal{X} = 2 \sum_{a=1}^{\mathrm{rk}\, {}^L\mathfrak{g}} d_a$$

The manifold

$$\mathcal{X}$$

is holomorphic symplectic. If it is smooth,

which is the case if the representations ${}^L\rho_i$ are minuscule,

it is also hyperkahler.

Our manifold $\mathcal{X}$ has several other useful descriptions.

The best known one is as an intersection

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu} = \mathrm{Gr}^{\vec{\mu}} \cap \mathrm{Gr}_{\nu}$$

of transversal slices in affine Grassmannian of G

$$\mathrm{Gr}_G = G((z))/G[z]$$

Here, the vector

$$\vec{\mu} = ({}^L w_1, \ldots, {}^L w_n)$$

encodes the singular monopole charges.

The description in terms of

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu} = \mathrm{Gr}^{\vec{\mu}} \cap \mathrm{Gr}_{\nu}$$

arises by thinking about singular G monopoles on

$$\mathbb{R} \times \mathbb{C}$$

as a sequence of Hecke modifications of holomorphic G-bundles on $\mathbb{C}$ parameterized by $\mathbb{R}$.

The loop variable z of the affine Grassmanian

$$\mathrm{Gr}_G = G((z))/G[z]$$

is the coordinate on $\mathbb{C}$

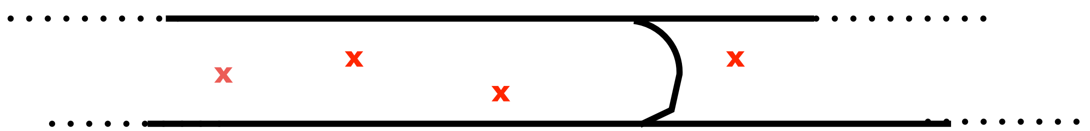
All the ingredients in

$$\Psi(a_1, \ldots, a_n) = \langle\lambda|\ \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n)\ |\lambda'\rangle$$

have a geometric interpretation in terms of $\mathcal{X}$,

starting with the (relative) positions of vertex operators on $\mathcal{A}$

$\mathcal{A}$



which are the complexified Kahler moduli of $\mathcal{X}$.

For quantum cohomology to be non-trivial,

one has to work equivariantly with respect to a torus action

$$\mathbb{C}^{\times}_{\mathbf{q}} \subset \mathrm{T}$$

that scales the holomorphic symplectic form.

This action acts on $\mathbb{C}$ in $\mathbb{R} \times \mathbb{C}$ by

$$z \to \mathbf{q} z$$

For it to be a symmetry, all the singular monopoles must be at the origin.

One works equivariantly with respect to a full torus of symmetries

$$\mathrm{T} = \Lambda \times \mathbb{C}^{\times}_{\mathfrak{q}}$$

The equivariant parameters for

$$\Lambda \subset \mathrm{T}$$

preserving the holomorphic symplectic form,

determine the highest weight vector of Verma module $\langle\lambda|$ in

$$\Psi(a_1, \ldots, a_n) = \langle\lambda|\, \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n)\, |\lambda'\rangle$$

The fact that Knizhnik-Zamolodchikov equation solved by

$$\Psi(a_1, \ldots, a_n) = \langle\lambda|\ \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n)\ |\lambda'\rangle$$

has a geometric interpretation as the

quantum differential equation of

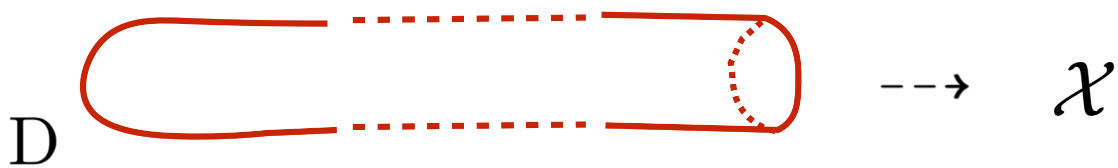
$$\mathcal{X}$$

computed by T- equivariant Gromov-Witten theory,

implies the conformal blocks too have a geometric interpretation.

Solutions of the quantum differential equation are

equivariant counts of holomorphic maps



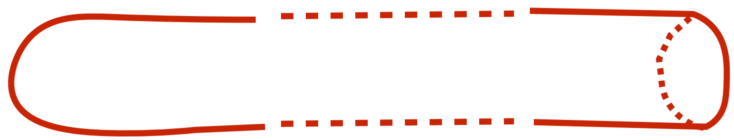
of all degrees computed by

equivariant Gromov-Witten theory.

These generating functions go under the name Givental's J-function

or, "cohomological Vertex function" of $\mathcal{X}$.

The domain curve D



is best thought of an infinite cigar with an S^1 boundary at infinity.

The boundary data is a choice of a K-theory class

$$[\mathcal{F}] \in K_{\mathrm{T}}(\mathcal{X})$$

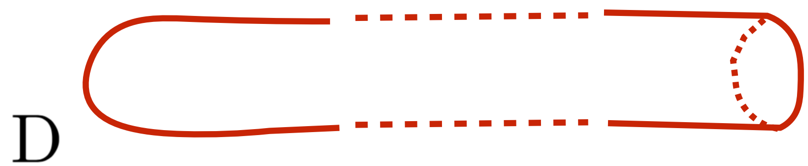
which determines which solution of the

Knizhnik-Zamolodchikov equation $\mathsf{Vertex}(\mathcal{F})$ computes.

The vertex function also depends on the insertion of a class in

$$H^*_{\mathrm{T}}(\mathcal{X})$$

at the origin of D .



In the present context, $H^*_{\mathrm{T}}(\mathcal{X})$ is identified with the weight space

$$({}^L\rho_1 \otimes \ldots {}^L\rho_\ell \otimes \ldots \otimes {}^L\rho_n)_\nu$$

of ${}^L\mathfrak{g}$ by

Geometric Satake correspondence.

The geometric interpretation of conformal blocks of

$$\widehat{L\mathfrak{g}}$$

in terms of

$$\mathcal{X}$$

has more information than the conformal blocks themselves.

Underlying the Gromov-Witten theory of

$$\mathcal{X}$$

is a two-dimensional supersymmetric sigma model with

$\mathcal{X}$ as a target space.

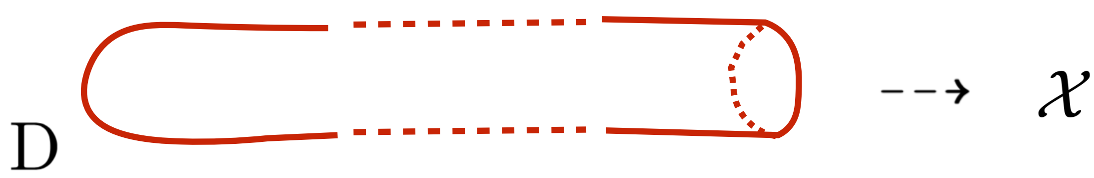
The physical meaning of

Gromov-Witten vertex function

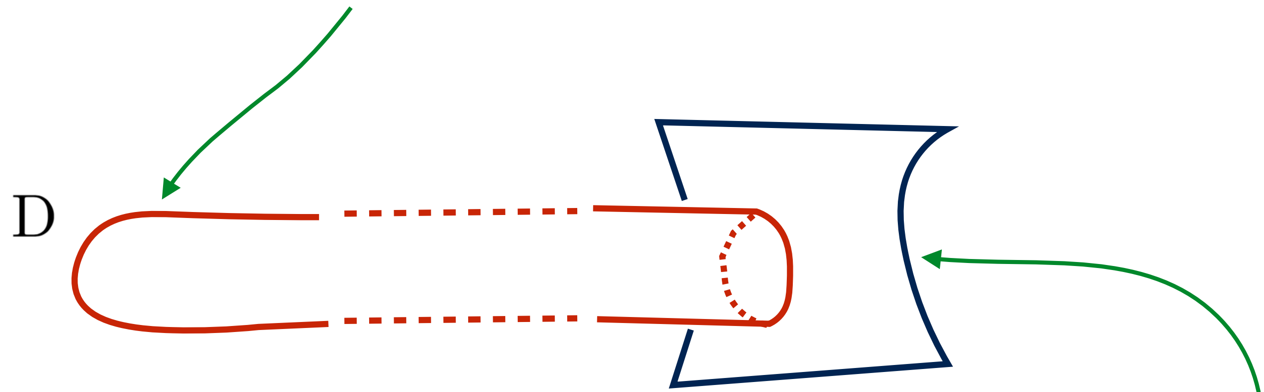
$$\mathsf{Vertex}(\mathcal{F})$$

is the partition function of the supersymmetric sigma model

with target $\mathcal{X}$ on D



One has, in the interior of D , an A-type twist



and at infinity, one places a B-type boundary condition.

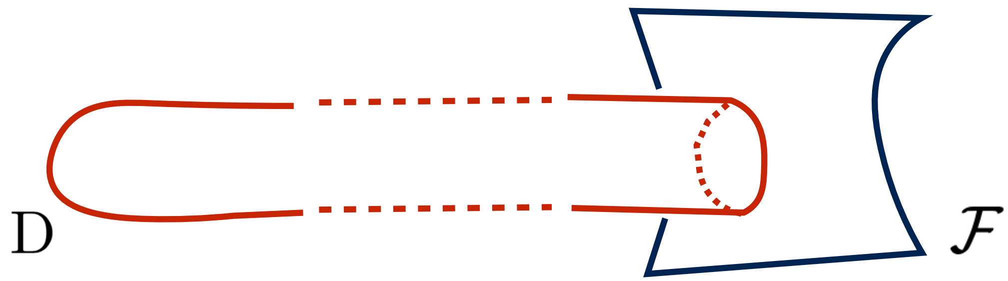
In our setting, the category of B-type boundary conditions is

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

the derived category of $\mathrm{T}-$equivariant coherent sheaves on $\mathcal{X}$

Picking, as a boundary condition, an object

$$\mathcal{F} \in D^b Coh_{\mathrm{T}}(\mathcal{X})$$



we get $\mathsf{Vertex}(\mathcal{F})$ as the partition function.

$\mathsf{Vertex}(\mathcal{F})$ depends on the choice of $\mathcal{F}$

only through its K-theory class

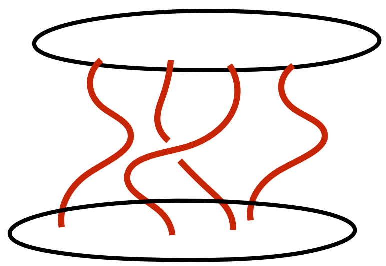
$$[\mathcal{F}] \in K_{\mathrm{T}}(\mathcal{X})$$

Since the Knizhnik-Zamolodchikov equation of

$$\widehat{{}^L\mathfrak{g}}$$

is the quantum differential equation of $\mathcal{X}$,

the action of $U_{\mathfrak{q}}({}^L\mathfrak{g})$ on the space of conformal blocks



B

is the monodromy of the quantum differential equation of $\mathcal{X}$,

along the path in its Kahler moduli corresponding to the braid B .

The action of monodromy on

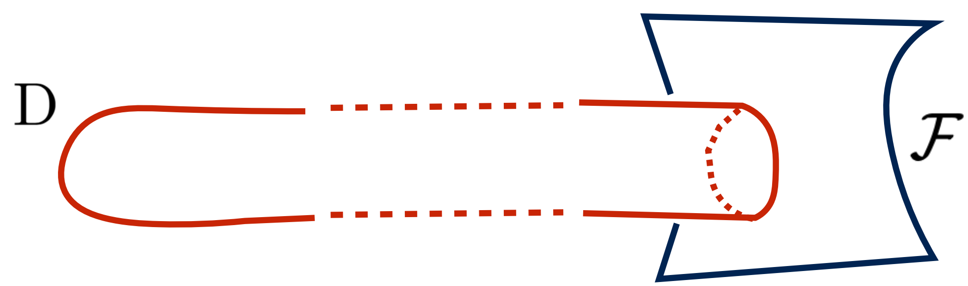
$$\text{Vertex}(\mathcal{F})$$

a-priori comes from its action on the K-theory class

$$[\mathcal{F}] \in K_{\mathrm{T}}(\mathcal{X})$$

of the brane at the boundary at infinity, although from perspective

of the 2d sigma model,



it really comes from the action on the brane $\mathcal{F} \in D^b Coh_{\mathrm{T}}(\mathcal{X})$ itself.

This is in fact a theorem, due to Bezrukavnikov and Okounkov, which states that,

for a class of holomorphic symplectic manifolds which includes our $\mathcal{X}$

the action of braiding on the K-theory

$$B: \ K_{\mathrm{T}}(\mathcal{X}) \to K_{\mathrm{T}}(\mathcal{X})$$

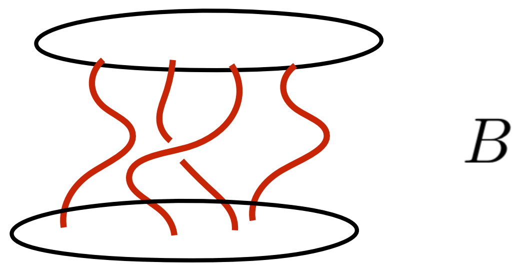
via the monodromy of the quantum differential equation

lifts to

a derived auto-equivalence functor of the category

$$\mathscr{B}: D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

Along a path B in Kahler moduli



the derived category stays the same, so the functor

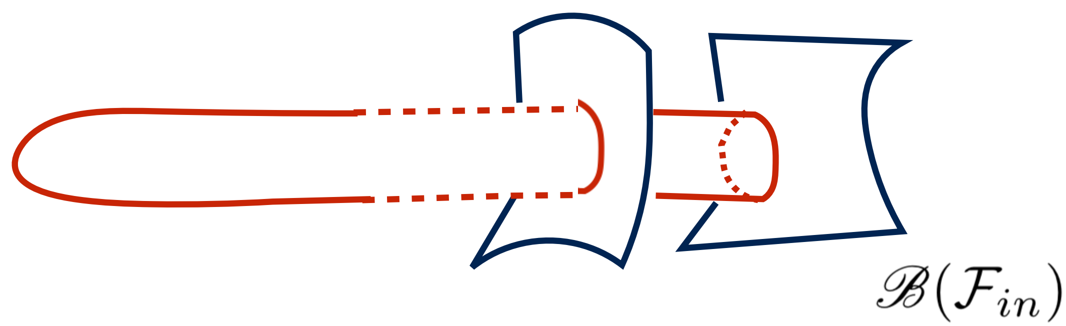
$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

is an equivalence of categories. Individual objects do change so we get a map

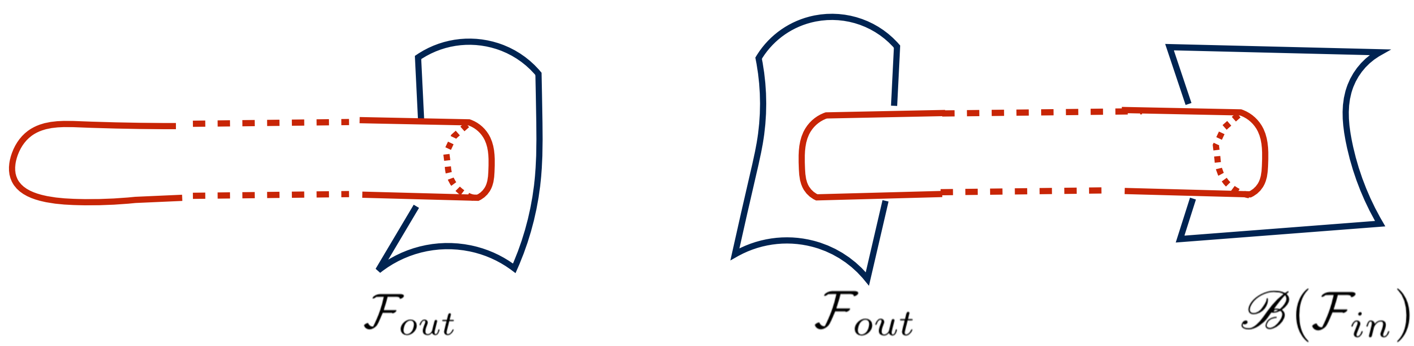
$$\mathcal{F} \to \mathscr{B}\mathcal{F}$$

that depends on the braid.

What this expresses physically is that we can cut the infinite cigar



very near the boundary, and insert a "complete set of branes",



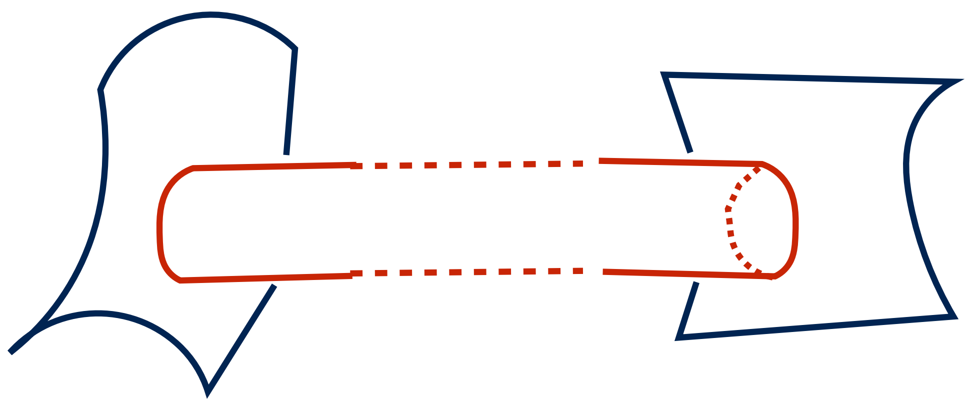
to extract matrix elements.

Existence of the complete set is part of of the conditions of the theorem.

The matrix element of the monodromy matrix

$$(\Psi_{\mathcal{F}_{out}} | \mathfrak{B} \, \Psi_{\mathcal{F}_{in}})$$

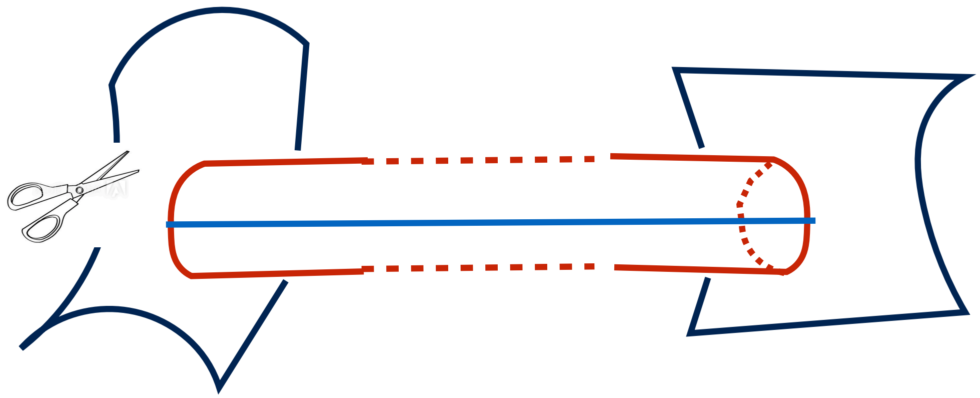
is the annulus amplitude of sigma model to



$\mathcal{F}_{out}$ $\mathscr{B}(\mathcal{F}_{in})$

with the pair of B-branes at the boundary.

The B-model annulus amplitude, is essentially per definition,



the supertrace, over the graded Hom space between the branes

$$\mathrm{Hom}^{*,*}(\mathcal{F}_{out}, \mathscr{B}\mathcal{F}_{in})$$

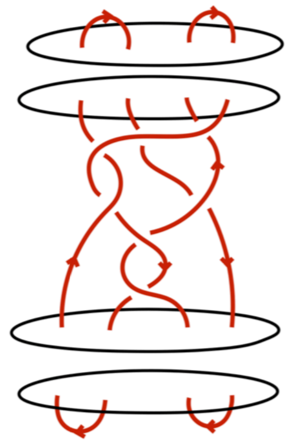
computed in

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

This also implies that quantum invariants of links

are categorified by

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$



since they too can be expressed as matrix elements of the braiding matrix

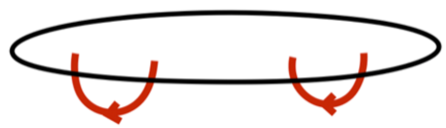
$$\left(\Psi_{\mathcal{F}_{out}} \middle| \mathfrak{B}\, \Psi_{\mathcal{F}_{in}}\right)$$

between pairs of conformal blocks.

For this one first needs to understand which objects of

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

correspond to conformal blocks

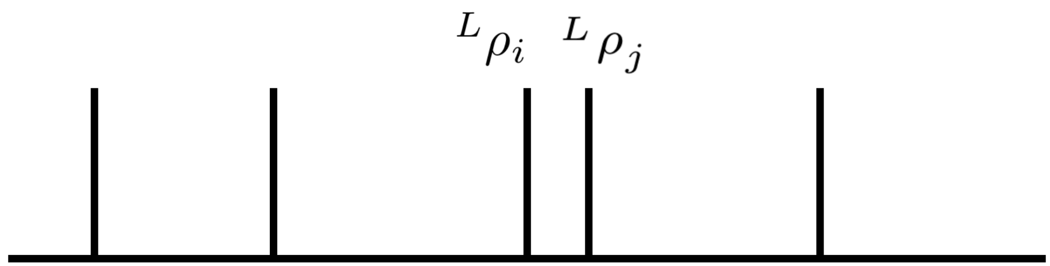


where pairs of vertex operators fuse to trivial representations.

This turns out not to be difficult.

$\mathcal{X}$ develops singularities.

as a pair of vertex operators approach each other



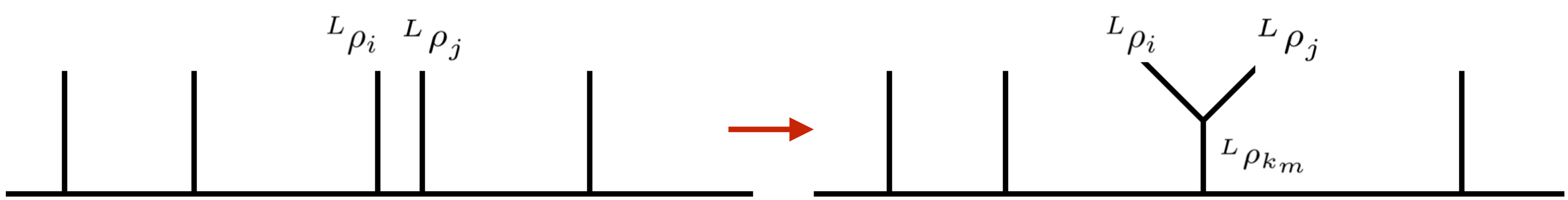
The singularities come from cycles that collapse as

$$|a_i| \to |a_j|$$

The collapsing cycles one gets turn out to be labeled by representations that occur in

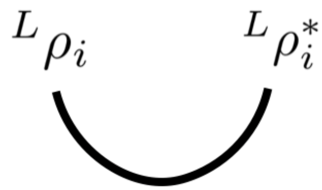
$$ {}^L\rho_i \otimes {}^L\rho_j = \bigotimes_{m=0}^{m_{max}} {}^L\rho_{k_m} $$

This is a reflection of the fact that as a pair of vertex operators approach, one gets a new natural basis of conformal blocks:



which are eigenvectors of braiding.

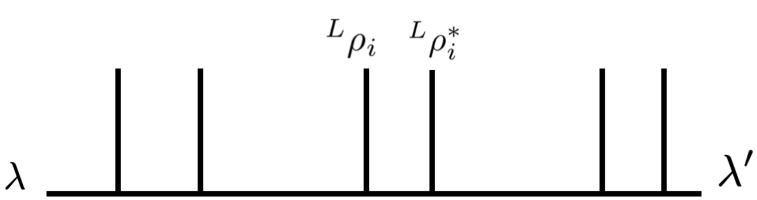
To a cap colored by representation ${}^L\rho_i$

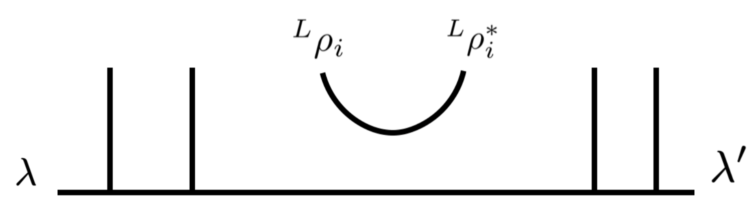


corresponds to a pair of vertex operators,

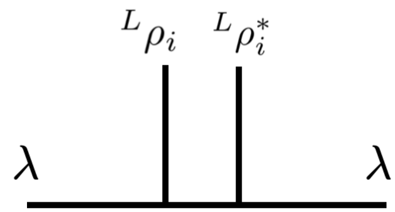
colored by conjugate representations

which approach each other and fuse to the identity:





For example, for a pair of conjugate minuscule representations

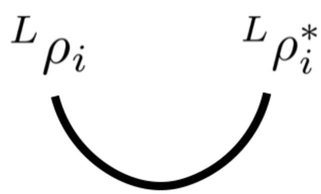


our manifold is

$$\mathcal{X} = T^* F_i = T^* G/P_i$$

where P_i is the corresponding maximal parabolic subgroup of G

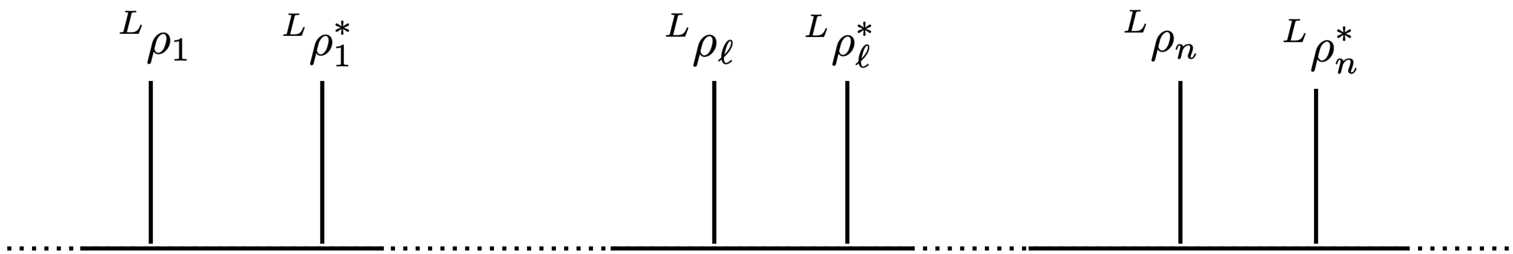
The object of $\mathcal{D}_{\mathcal{X}}$ corresponding to



is the structure sheaf $\mathcal{F}_i = \mathcal{O}_{F_i}$ of F_i .

More generally, to a collection of complex conjugate pairs of

minuscule representations



corresponds a local neighborhood of

$$\mathcal{X} = \mathrm{Gr}_0^{\vec{\mu}}$$

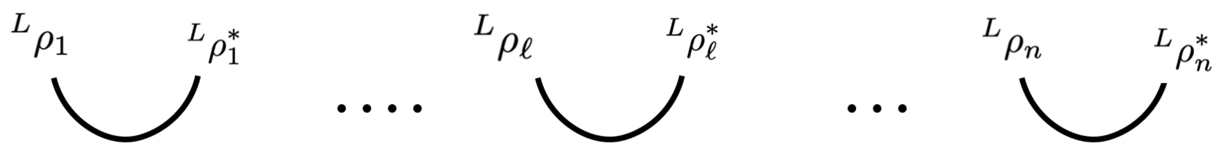
where we can approximate it as

$$\mathcal{X} \sim T^* F$$

where

$$F = G/P_1 \times \ldots \times G/P_\ell \times \ldots \times G/P_n$$

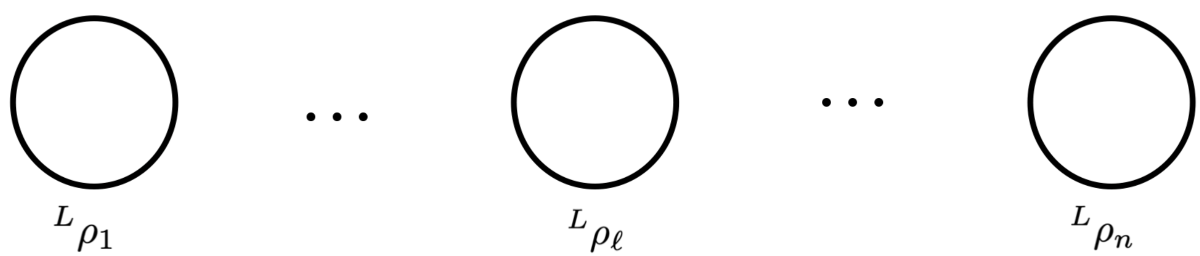
Object of $\mathscr{D}_{\mathcal{X}}$ which corresponds to a collection of caps



is the structure sheaf $\mathcal{F} = \mathcal{O}_F$ of F in

$$\mathcal{X} = \mathrm{Gr}_0^{\vec{\mu}}$$

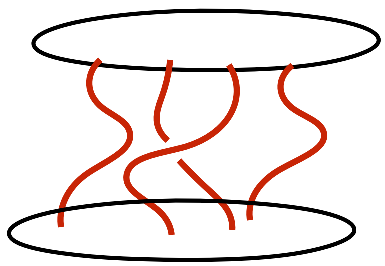
In particular, $\mathrm{Hom}^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{F})$ categorifies



One can also get a fairly detailed picture of the functors

$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

that categorify braiding



B

By it origin in the sigma model to $\mathcal{X}$, the functor

$$\mathscr{B} : D^b Coh_{\mathrm{T}}(\mathcal{X}) \to D^b Coh_{\mathrm{T}}(\mathcal{X})$$

comes from a variation of stability condition on

$$D^b Coh_{\mathrm{T}}(\mathcal{X})$$

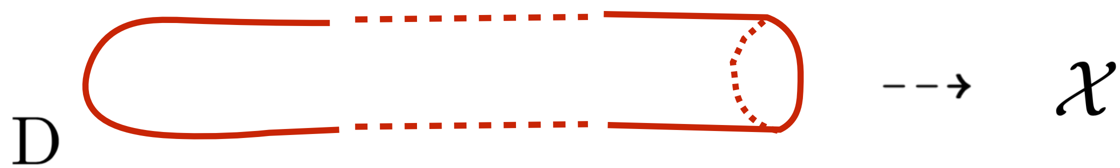
defined with respect to a central charge function

$$\mathcal{Z}^0(\mathcal{F}) : K(\mathcal{X}) \to \mathbb{C}$$

which is a close cousin of $\mathsf{Vertex}(\mathcal{F})$

Like Vertex($\mathcal{F}$) , it can be computed by

equivariant Gromov-Witten theory



except with trivial insertion at the origin.

In the end one turns off the equivariant parameters,

to get a map

$$\mathcal{Z}^0(\mathcal{F}) : K(\mathcal{X}) \to \mathbb{C}$$

depending only on Kahler moduli.

The stability condition defined with respect to

$$\mathcal{Z}^0(\mathcal{F}) : K(\mathcal{X}) \to \mathbb{C}$$

is known as the Pi stability condition,

discovered by Douglas.

Our setting should provide a model example of
a Bridgeland stability condition on

$$D^b Coh_{\mathrm{T}}(\mathcal{X})$$

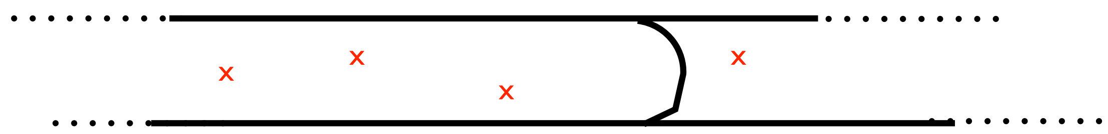
Since $\mathcal{X}$ is also hyper-Kahler, the stability

structure which comes from

$$\mathcal{Z}^0(\mathcal{F}) : K(\mathcal{X}) \to \mathbb{C}$$

is extremely simple.

It is constant in a chamber in Kahler moduli which



corresponds to fixing the order of vertex operators,

and only changes when a pair of them trade places.

$$\Phi_{L_{\rho_i}}(a_i) \otimes \Phi_{L_{\rho_j}}(a_j) \quad \longrightarrow \quad \Phi_{L_{\rho_j}}(a_j) \otimes \Phi_{L_{\rho_i}}(a_i)$$

Near a wall in Kahler moduli where

$$a_i \to a_j$$

we get vanishing cycles F_k^{ij} corresponding to ways of fusing:

$$\Phi_{{}^L\rho_i}(a_i) \otimes \Phi_{{}^L\rho_j}(a_j) \xrightarrow{a_i \to a_j} \Phi_{{}^L\rho_k}(a_j)$$

and objects $\mathcal{F}_k \in \mathscr{D}_{\mathcal{X}}$ whose central charge vanishes as fast as the dimension of the cycle

$$\mathcal{Z}_k^0 = \mathcal{Z}^0(\mathcal{F}_k) \sim (a_i - a_j)^{\dim F_k^{ij}} \times \text{finite}$$

This leads to a filtration on the derived category,

by the order of vanishing of $\mathcal{Z}^0$

In fact, one gets a pair of such filtration, one on each side of the wall:

$$\mathscr{D}_{k_0} \subset \mathscr{D}_{k_1} \ldots \subset \mathscr{D}_{k_{max}} = \mathscr{D}_{\mathcal{X}}$$

for $|a_i| < |a_j|$ and

$$\mathscr{D}'_{k_0} \subset \mathscr{D}'_{k_1} \ldots \subset \mathscr{D}'_{k_{max}} = \mathscr{D}'_{\mathcal{X}}$$

for $|a_j| < |a_i|$.

Crossing the wall preserves the filtrations,

since it has the effect of mixing up objects of a given order of vanishing,

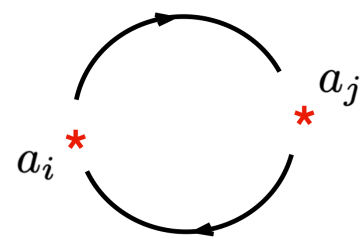
with those of that vanish faster, and which belong to lower orders

in the filtration.

The derived equivalence functor is a degree shift acting on

$$\mathscr{B} : \mathscr{D}_{k_m}/\mathscr{D}_{k_{m-1}} \to \mathscr{D}'_{k_m}/\mathscr{D}'_{k_{m-1}}$$

which depends only on the order in the filtration,



and on the path around the singularity

and which comes from the equivariant central charge.

$$\mathcal{Z}_{k_m} \longrightarrow (-1)^{\pi i D_m}\, \mathfrak{q}^{-T_m/2}\, \mathcal{Z}_{k_m}$$

This provides an example of a perverse equivalence of

Rouquier and Chuang,

which comes from geometry and physics.

Mirror symmetry

gives a second description of homological

knot invariants.

It is based on the "equivariant mirror" of

$$\mathcal{X}$$

The equivariant mirror

is a Landau-Ginzburg theory with target Y ,

and potential W_{LG} .

Ordinary, non-equivariant mirror of

$$\mathcal{X}$$

is a hyper-Kahler manifold

$$\mathcal{Y}$$

which is, to a first approximation,

given by a hyper-Kahler rotation of $\mathcal{X}$

As $\mathcal{X}$ has only Kahler but not complex moduli,

due to the T- equivariance we impose,

$\mathcal{Y}$ has only complex but no Kahler moduli turned on.

A description based on

$$\mathcal{Y}$$

would give a symplectic geometry approach to the categorification problem,

with

$$\mathscr{D}_{\mathcal{X}} = D^b Coh_{\mathrm{T}}(\mathcal{X})$$

replaced by its homological mirror, an appropriate category of

Lagrangian branes on $\mathcal{Y}$

At the moment, one does not have this,

since it is not known how to

describe the mirror of the

$$\mathbb{C}^\times_{\mathfrak{q}} \subset \mathrm{T}$$

action. Without it, one gets “symplectic” homological link invariants,

which describe the theory at

$$\mathfrak{q} = 1$$

such as those in the work of Seidel and Smith, for Khovanov homology.

There is an alternative symplectic geometry approach,

where the dependence of the theory

on $\mathfrak{q} = e^{2\pi i\beta}$

instead of being mysterious,

is manifest.

The key fact is that, since we work equivariantly with respect to the

$$\mathbb{C}^\times_{\mathfrak{q}} \subset \mathrm{T}$$

action on $\mathcal{X}$, which scales the holomorphic symplectic form

all the relevant information about its geometry

is contained in its fixed locus,

$$X = \mathcal{X}|_{\mathbb{C}^\times_{\mathfrak{q}}}$$

which is a holomorphic Lagrangian, its “core”.

Instead of working with $\mathcal{X}$ and its mirror $\mathcal{Y}$

one can work with the the core X and the core's mirror Y

$$\begin{array}{ccc} \mathcal{X} & \xleftrightarrow{\text{mirror}} & \mathcal{Y} \\ \uparrow & & \downarrow \\ X & \xleftrightarrow{\text{mirror}} & Y \end{array}$$

Working equivariantly with respect to $\mathbb{C}^{\times}_{\mathfrak{q}} \subset \mathrm{T}$ action on $\mathcal{X}$,

the bottom row has as much information about the geometry

as the top.

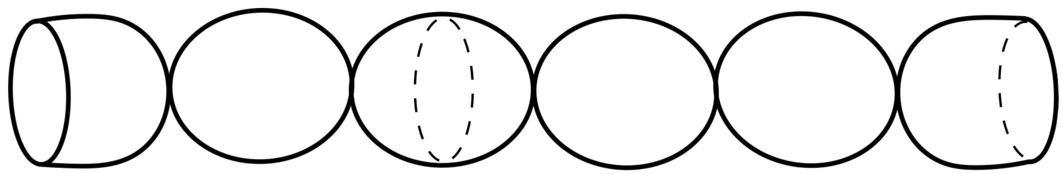
While X embeds into $\mathcal{X}$ as a

holomorphic Lagrangian submanifold of dimension n

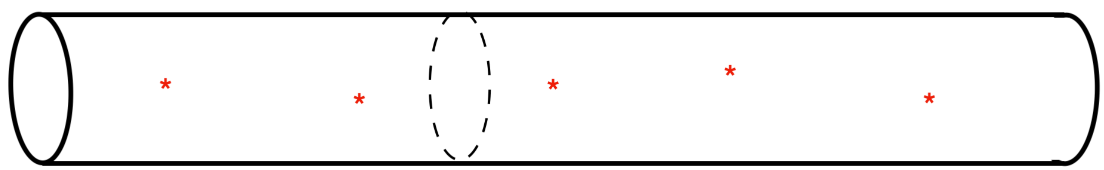
$\mathcal{Y}$ fibers over Y with holomorphic Lagrangian $(\mathbb{C}^{\times})^n$ fibers

For example, for

$\mathcal{X}$ which is an A_n surface, its core X looks like



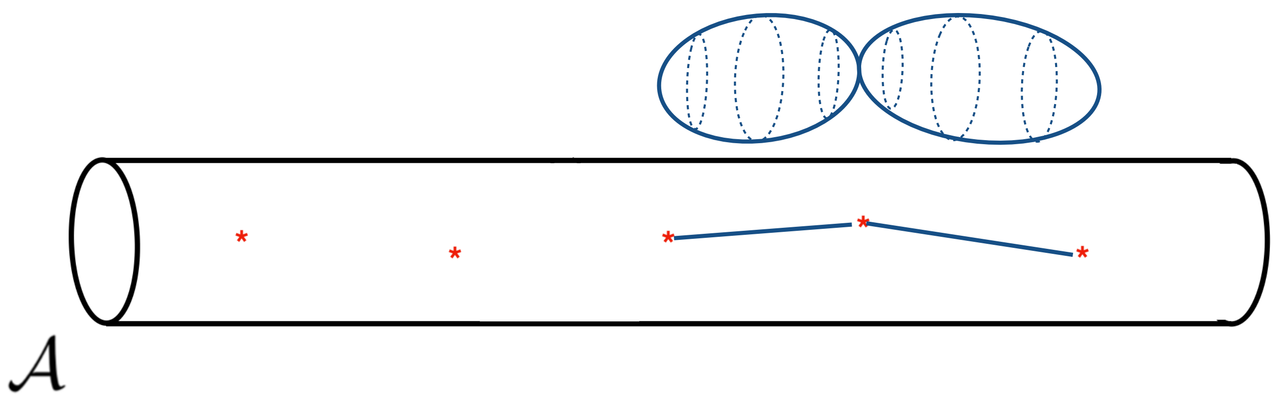
and its mirror $\mathcal{Y}$ is a $\mathbb{C}^\times$ fibration over Y, which looks like



and is mirror to X.

In this case, Y is a single copy of the surface $\mathcal{A}$ from the from the beginning of the talk, with marked points where the $\mathbb{C}^\times$ fibration degenerates.

Mirror to vanishing $\mathbb{P}^1$'s in X



are Lagrangians in Y that begin and end at the punctures, and which are projections from Lagrangian spheres in $\mathcal{Y}$.

Conjecturally, the equivariant mirror of

$$\mathcal{X} = \mathrm{Gr}_{\nu}^{\vec{\mu}}$$

and the ordinary mirror of its core X,

is a Landau-Ginsburg model, with target

$$Y = (\mathcal{A})^{\mathrm{rk},*}/\mathrm{Weyl}$$

where $d = \dim_{\mathbb{C}} X$, and specific potential

$$W_{LG}$$

mirror to the equivariant $\mathrm{T}-$ action.

The potential W_{LG} is a multi-valued function on Y ,

which is a sum of three types of terms, all coming from the equivariant actions

a term coming from the $\Lambda \subset \mathrm{T}$ -action:

$$W_1 = \sum_a \sum_\alpha (\lambda, {}^L e_a) \ln(x_{\alpha,a})$$

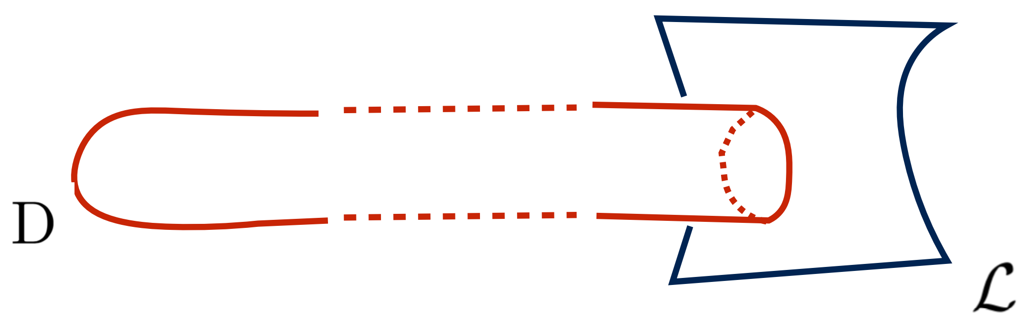
and two which come from the $\mathbb{C}^\times_{\mathfrak{q}} \subset \mathrm{T}$ action:

$$W_2 = \beta \sum_a \sum_{\alpha,i} \ln(x_{\alpha,a} - a_i)^{-({}^L e_a, {}^L w_i)} \quad , \quad W_3 = \beta \sum_{a,b} \sum_{\alpha<\beta} \ln(x_{\alpha,a} - x_{\beta,b})^{-({}^L e_a, {}^L e_b)}$$

Mirror symmetry predicts that the conformal block of $\widehat{{}^L\mathfrak{g}}$

$$\langle\lambda|\,\Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\,|\lambda'\rangle$$

is the partition function of the B-twisted theory on D ,



with A-type boundary condition at infinity, corresponding to the Lagrangian $\mathcal{L}$ in Y .

Such amplitudes have the following form

$$\int_L \Phi_a \, \Omega \, e^{W_{LG}}$$

where Ω is the top holomorphic form on Y ,

W_{LG}

is the Landau-Ginsburg potential,

and Φ 's are the chiral ring operators.

We are rediscovering here, from mirror symmetry,

the integral formulation of the $\widehat{{}^L\mathfrak{g}}$ conformal blocks

$$\langle\lambda|\, \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\, |\lambda'\rangle$$

which goes back to work of Feigin and E.Frenkel in the '80's

and Schechtman and Varchenko.

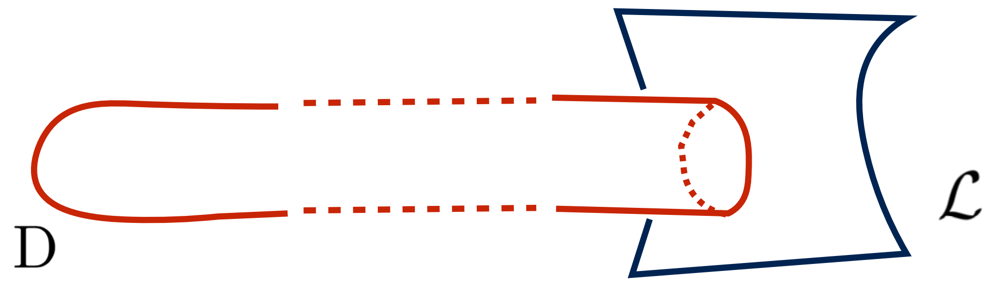
There is a reconstruction theory,
due to Givental and Teleman,
which says that starting with the genus zero data,
or more precisely, with the solution of quantum differential equation,
one gets to reconstruct all genus topological string amplitudes
of a semi-simple 2d field theory.

Thus, the B-twisted the Landau-Ginsburg model (Y, W_{LG})
and A-twisted sigma model on $\mathcal{X}$,
working equivariantly with respect to T ,
are expected to be equivalent to all genus.

Corresponding to a solution of the

Knizhnik-Zamolodchikov equation

is an A-brane at the boundary of D at infinity,



The brane is an object of

$$\mathscr{D}_Y = D(\mathcal{FS}(Y, W_{LG}))$$

the derived Fukaya-Seidel category of A-branes on Y with potential W_{LG}

In general, to formulate a category of

A-branes on a non-compact manifold such as

$$Y$$

requires work, to cure the non-compactness.

In the present case, we are after a symplectic-geometry based approach to knot homology.

The Lagrangians we need are all compact,

since they are related by mirror symmetry

to compact vanishing cycles on $\mathcal{X}$.

For such Lagrangians,

there are no issues with non-compactness of Y

The superpotential W_{LG} would have played no role either,

were it single valued.

W_{LG} is not single valued for us,

but its only effect is to provide additional gradings on

$$HF^{*,*}(L_{out}, L_{in}) = \bigoplus_{n\in\mathbb{Z}, k\in\mathbb{Z}^{\mathrm{rkT}}} Hom_{\mathscr{D}_Y}(L_{out}, L_{in}[n]\{k\})$$

the Floer cohomology groups.

The additional grades may be defined

by lifting the phase of

$$\Omega e^{W_{LG}}$$

to a real valued function on the Lagrangian,

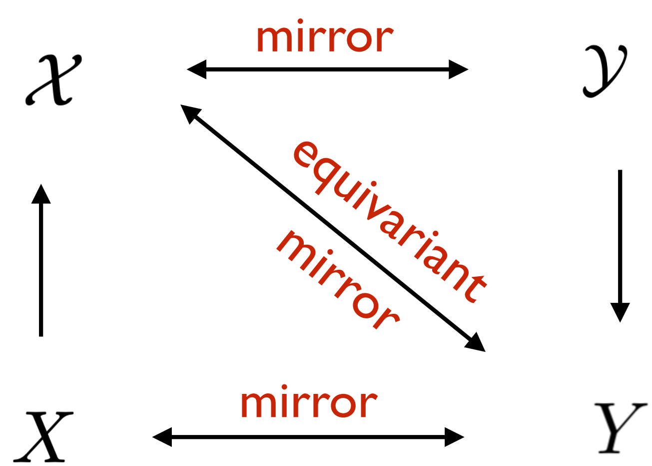
analogously to the way the lift of the phase of

$$\Omega$$

is used used to define Maslov grading.

Mirror symmetry

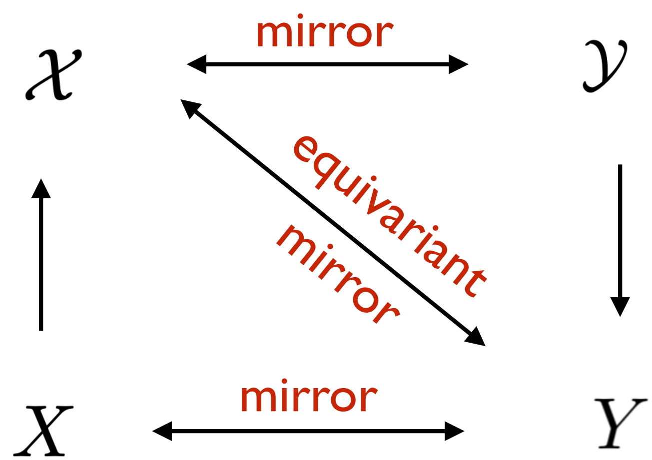
helps us understand exactly which questions we need to ask



to recover homological knot invariants from Y .

Since Y is an ordinary mirror of X,

we should start by understanding how to recover



homological knot invariants from X, instead of $\mathcal{X}$

Every B-brane on $\mathcal{X}$ which is relevant to us

"comes from" a B-brane on X

via a pushforward functor,

$$f_* : \mathscr{D}_X \to \mathscr{D}_{\mathcal{X}}$$

that interprets a sheaf F on X, (more precisely, an object of $\mathscr{D}_X$)

as a sheaf $\mathcal{F} = f_* F$ on $\mathcal{X}$

This functor has an adjoint, that goes the other way,

$$f^* : \mathscr{D}_{\mathcal{X}} \to \mathscr{D}_X$$

that takes a sheaf on $\mathcal{X}$ to a sheaf on X ,

by tensoring with the structure sheaf $\mathcal{O}_X$, and restricting to X

The fact these are adjoint functors is what lets us relate the computations of

Hom's on $\mathcal{X}$ to those on X .

Given any pair of objects on $\mathcal{X}$ that come from X

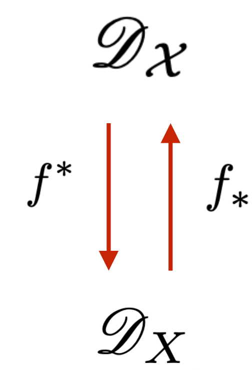
the Hom between them, computed upstairs, in $\mathscr{D}_{\mathcal{X}}$

$$\mathcal{F} = f_* F, \;\; \mathcal{G} = f_* G$$

agrees with the Hom downstairs, in $\mathscr{D}_X$,

$$Hom^{*,*}_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom^{*,*}_{\mathscr{D}_X}(f^* f_* F, G)$$

after replacing F with $f^* f_* F$



By mirror symmetry, for every pair of objects

$$\mathcal{F} = f_* F, \;\; \mathcal{G} = f_* G$$

on $\mathcal{X}$ which come from X, there is a pair of Lagrangians

$$L_F \; , \quad L_G$$

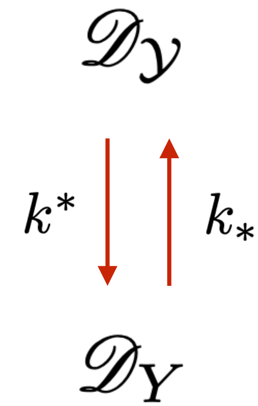
on Y which are mirror to F and G ,

$$Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F}, \mathcal{G}) = Hom_{\mathscr{D}_{(Y,W)}}(k^* k_* L_F, L_G)$$

such that Hom's on Y agree with those on $\mathcal{X}$.

The functors k_* and k^* that enter

$$Hom_{\mathscr{D}_{\mathcal{X}}}(\mathcal{F},\mathcal{G}) = Hom_{\mathscr{D}_{(Y,W)}}(k^* k_* L_F, L_G)$$



relate objects on Y and on $\mathcal{Y}$,

in a way that mirrors f^* and f_* ,

They come from a Lagrangian correspondence on $Y^- \times \mathcal{Y}$

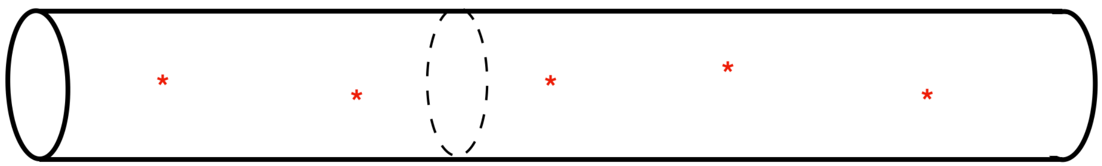
The construction of these functors,

and the parallel understanding of mirror symmetries upstairs and downstairs

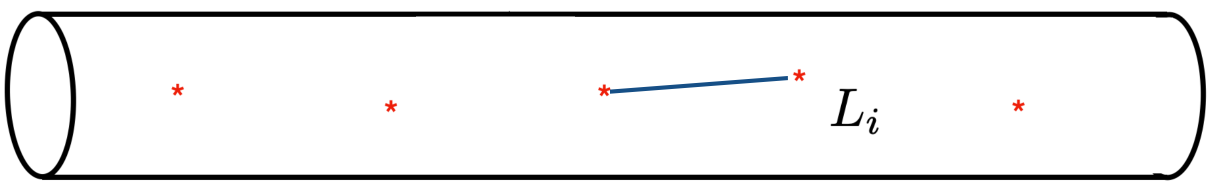
is joint work with Shende and McBreen.

Recall our example, Y the equivariant mirror to

$\mathcal{X}$ which is the A_n surface.



Mirror to i-th vanishing $\mathbb{P}^1$ in X is the Lagrangian



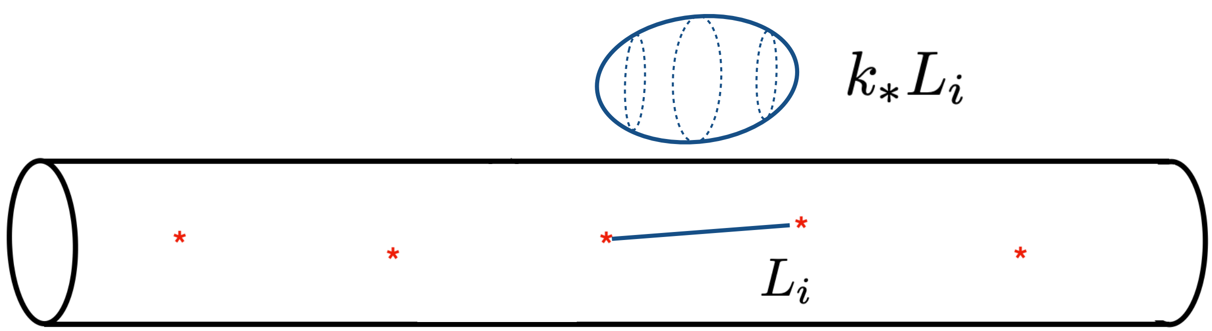
The mirror of $\mathcal{X}$ is $\mathcal{Y}$ the multiplicative A_n surface,

which is a $\mathbb{C}^\times$ fibration over Y.

The functor $k_* : \mathscr{D}_Y \to \mathscr{D}_\mathcal{Y}$ maps any Lagrangian in Y,

to a Lagrangian in $\mathcal{Y}$, which fibers over Y with S^1 fibers.

In particular, $k_* L_i$ is the i-th vanishing sphere in $\mathcal{Y}$.



The functor going the other way

$$k^* : \mathscr{D}_{\mathcal{Y}} \to \mathscr{D}_Y$$

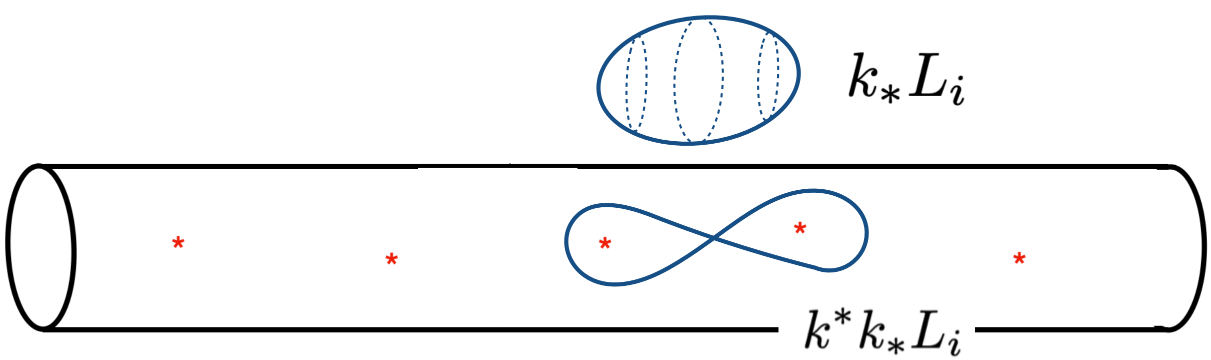
does not send the vanishing sphere $k_* L_i$ back to L_i :

$$k^* k_* L_i \neq L_i$$

Instead, either computing it either from mirror symmetry,

or its via its definition coming from a Lagrangian correspondence,

one finds a figure eight Lagrangian

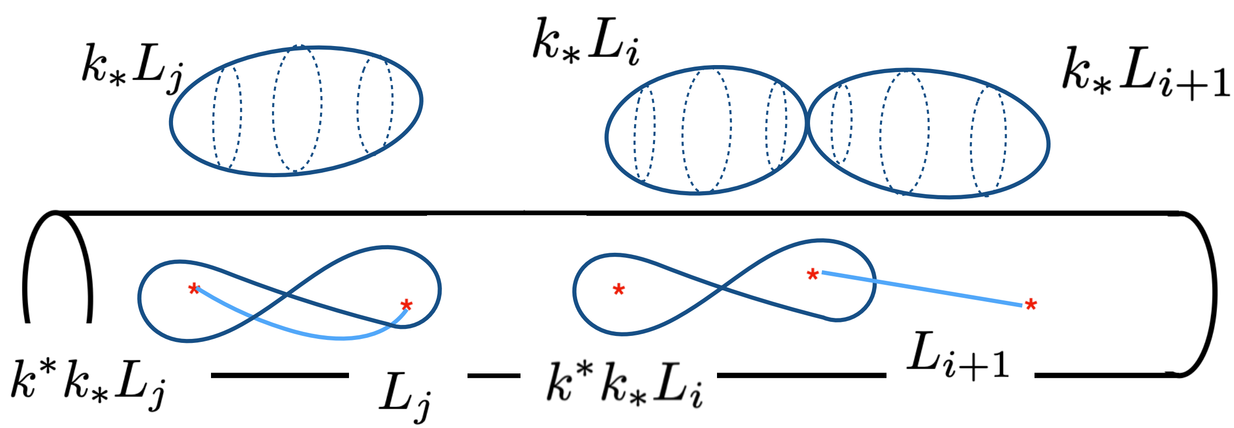


The basic virtue of the pair of adjoint functors,

is that one ends up preserving Hom's.

$$Hom_{\mathscr{D}_{\mathcal{Y}}}(k_* L_i, k_* L_j) = Hom_{\mathscr{D}_Y}(k^* k_* L_i, L_j)$$

It is not difficult to see that this indeed is the case



The example we just gave is relevant construction of

Khovanov homology,

since the needed $\mathcal{Y}$ can be described as

$$\mathcal{Y} = Sym^m(A_{2m-1})^*$$

an open subspace in the symmetric product of

m copies of an A_{2m-1} surface:

$$uv = \prod_{I=1}^{2m}(x - a_I) \qquad u, v \in \mathbb{C}, \quad x \in \mathbb{C}^\times$$

This

$$\mathcal{Y} = Sym^m(A_{2m-1})^*$$

is the same geometry Seidel and Smith studied

in their work on symplectic Khovanov homology,

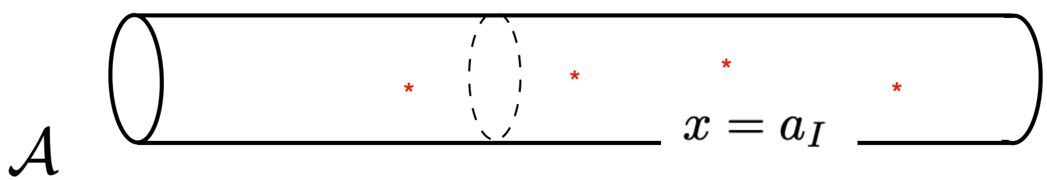
as shown by Manolescu.

The corresponding Landau-Ginsburg model

has the target which is also an open subset of symmetric product,

$$Y = Sym^m(\mathcal{A})^*$$

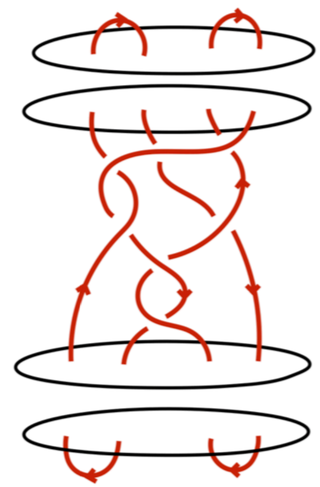
of the surface where the conformal blocks live



with potential

$$W_{LG} = \sum_{i=1}^{m} [\lambda \ln(x_i) - \beta \sum_{I=1}^{2m} \ln(x_i - a_I) + \beta \sum_{i \neq j} \ln(x_i - x_j)]$$

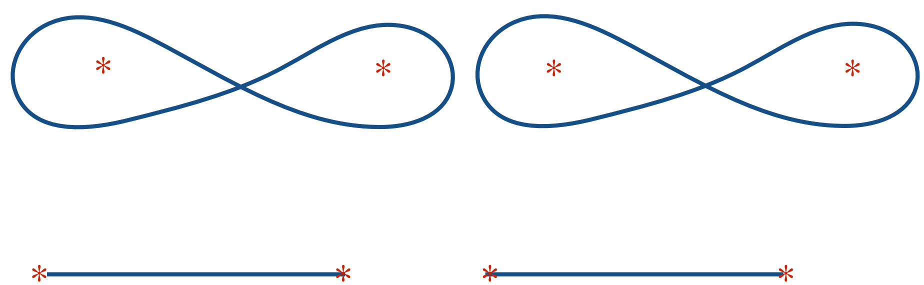
The objects corresponding to top and the bottom



are the Lagrangians:

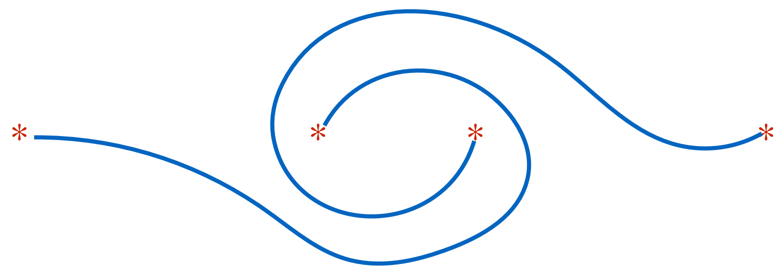
L_{out}

L_{in}



To get a non-trivial link, one starts by transporting L_{in} along the braid:

$\mathscr{B}L_{in}$



The generators of the Floor co-chain complex are the intersection points

$$p \in L_{out} \cap \mathscr{B}L_{in}$$

graded by the Maslov index, and the new grading that comes

from the non-single valued super-potential.

The homological link invariant is the Floer cohomology group

$$HF^{*,*}(L_{out}, \mathscr{B}L_{in}) = \bigoplus_{n\in\mathbb{Z}, k\in\mathbb{Z}^{\mathrm{rkT}}} Hom_{\mathscr{D}_Y}(L_{out}, \mathscr{B}L_{in}[n]\{k\})$$

whose differential is obtained by counting holomorphic disks

of Maslov index one on Y , as in Floer's theory.

The condition that the disk is in Y

requires W_{LG} to be single valued around the boundary of the disk,

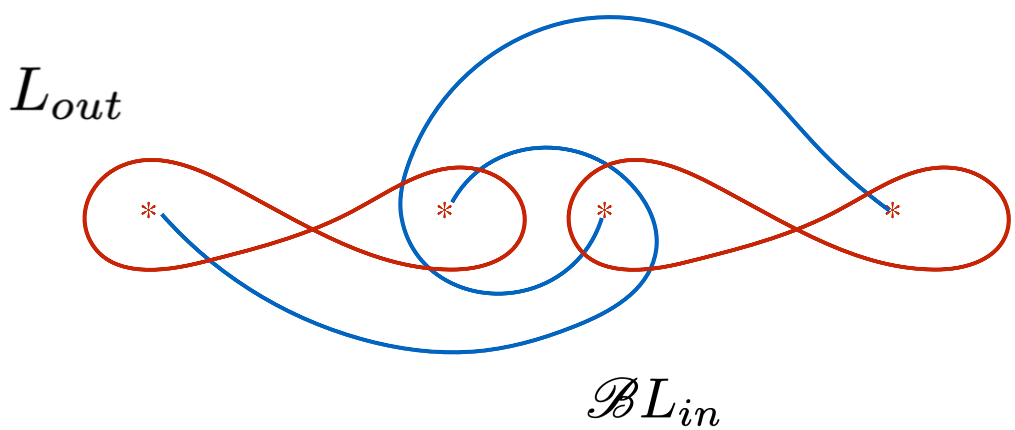
so equivariant grade of the differential is zero.

To prove this categorifies the Jones polynomial is easy.

To compute the Jones polynomial,

one simply counts the intersection points keeping track of gradings.

$$(\Psi_{out}, B\Psi_{in}) = \sum_{\mathcal{P} \in L_{out} \cap \mathscr{B} L_{in}} (-1)^{F(\mathcal{P})} \, \mathfrak{q}^{J(\mathcal{P})}$$

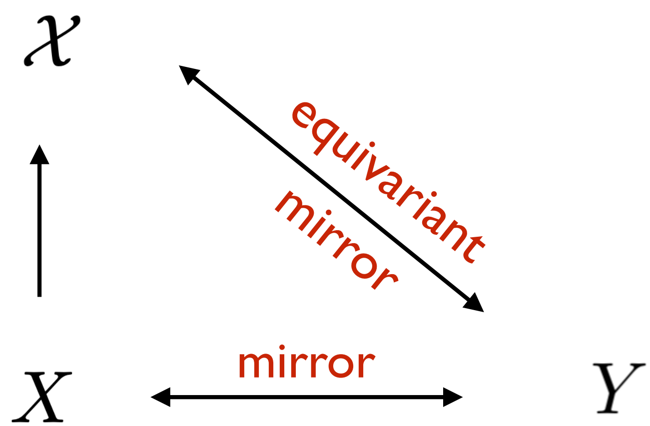


The result is the construction of Jones polynomial due to Bigelow.

The theory that results is no doubt novel and somewhat subtle.

However all of its features are forced on us by

by equivariant homological mirror symmetry,



In particular, a failure of its existence would be a failure of mirror symmetry.

In the remaining time,

let me try to explain the string theory origin of this construction.

The two dimensional theories we have been

discussing originate directly from string theory.

A helpful observation is another interpretation of

$$\mathcal{X}$$

In addition to being the intersection of slices in the affine Grassmannian

$$\mathcal{X} = {\rm Gr}^{\vec{\mu}}{}_{\nu} = {\rm Gr}^{\vec{\mu}} \cap {\rm Gr}_{\nu}$$

and the moduli space of singular G -monopoles,

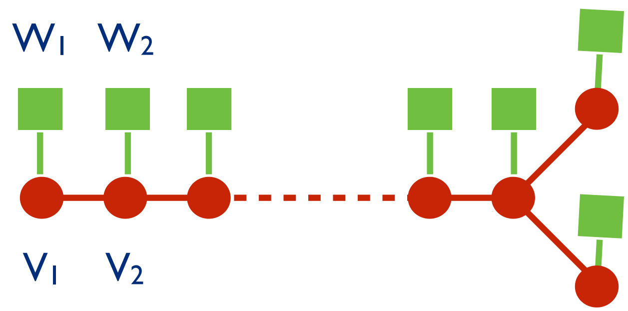
$\mathcal{X}$ is also a Coulomb branch of a three dimensional gauge theory.

The theory is a

three dimensional quiver gauge theory

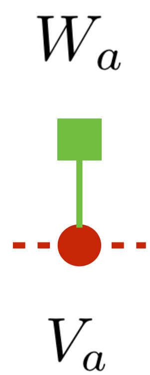
with quiver $\mathcal{Q}$

$\mathfrak{g} = D_n$



based on the Dynkin diagram of $\mathfrak{g}$

The ranks of the vector spaces



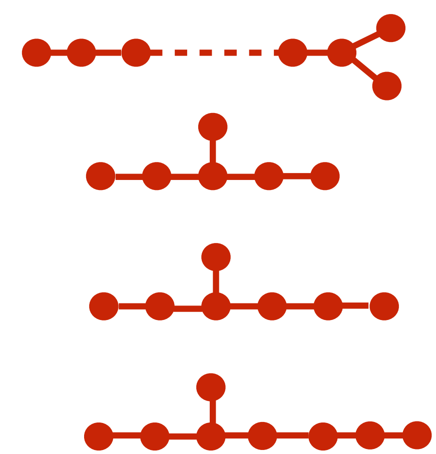
are determined from $\mu = |\vec{\mu}|,\ \nu$ in

$$\mathcal{X} = \mathrm{Gr}^{\vec{\mu}}{}_{\nu} = \mathrm{Gr}^{\vec{\mu}} \cap \mathrm{Gr}_{\nu}$$

This gauge theory arises on defects, or more precisely, on D-branes

of a certain six dimensional "little" string theory

labeled by a simply laced Lie algebra $\mathfrak{g}$



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

with (2,0) supersymmetry.

The six dimensional string theory is obtained by taking a limit of IIB string theory on an ADE surface singularity of type

$$\mathfrak{g}$$

In the limit, one keeps only the degrees of freedom supported at the singularity and decouples the 10d bulk.

One wants to study the six dimensional (2,0) little string theory on

$$M_6 \approx \mathcal{A} \times D \times \mathbb{C}$$

where

$$\mathcal{A} = S^1 \times \mathbb{R}$$

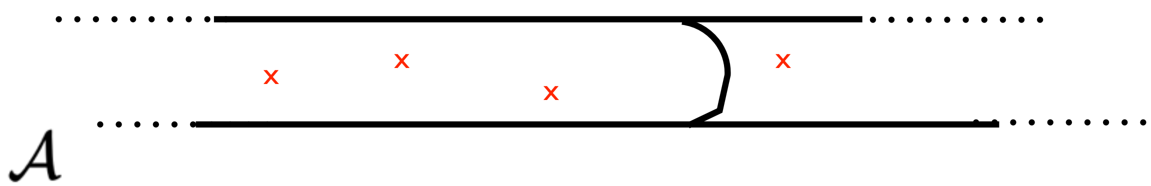
is the Riemann surface where the conformal blocks live,

$\mathcal{A}$ 

$\mathbb{R} \times \mathbb{C}$ is the space where the monopoles live

and D is the domain curve of the 2d theories we had so far.

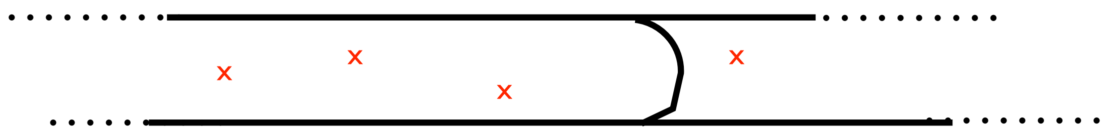
The vertex operators on the Riemann surface



come from a collection of defects in the little string theory,

which are inherited from D-branes of the ten dimensional string.

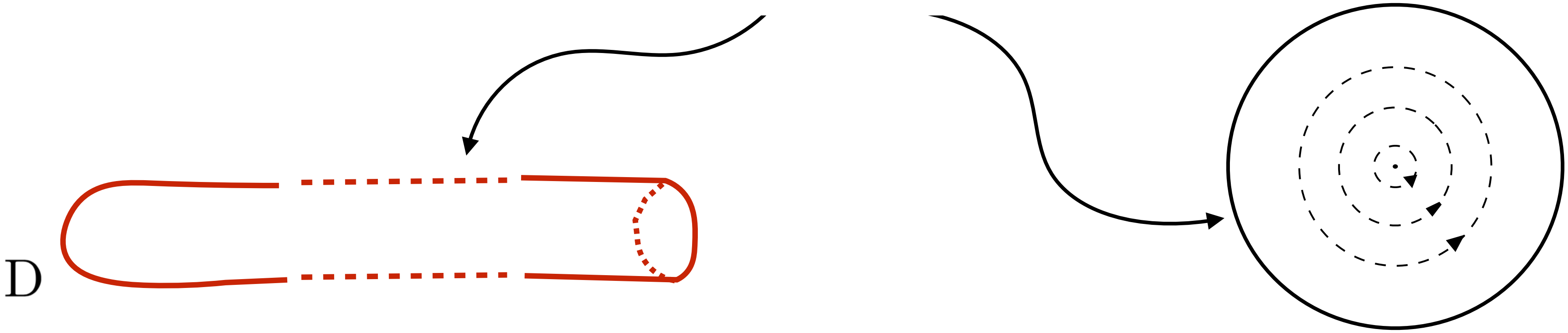
The D-branes needed are



$\mathcal{A}$

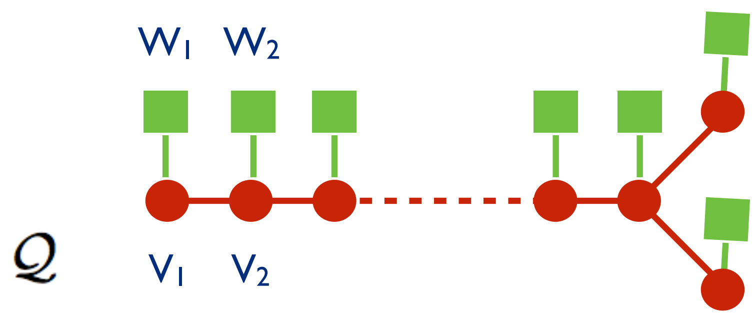
two dimensional defects of the six dimensional theory on

$$M_6 \approx \mathcal{A} \times D \times \mathbb{C}$$



supported on D and the origin of $\mathbb{C}$

The theory on the D-branes is the quiver gauge theory



This is a consequence of the familiar description of D-branes on ADE singularities due to Douglas and Moore in '96.

The theory on the D-branes supported on D

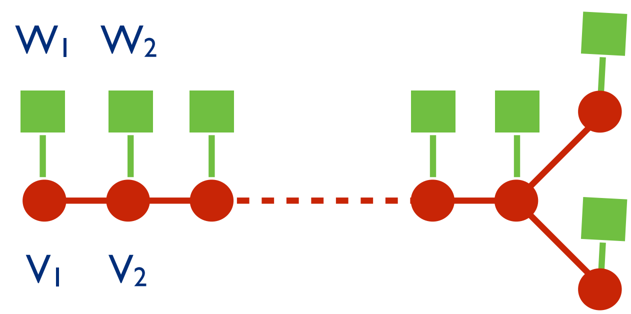
is a three dimensional quiver gauge theory on

$$\mathrm{D} \times S^1$$

rather than a two dimensional theory on D ,

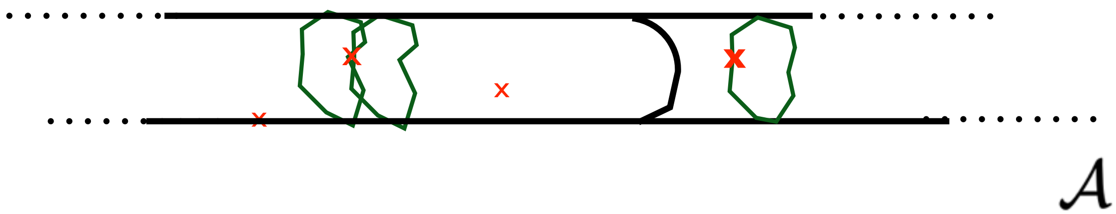
due to a stringy effect.

$\mathcal{Q}$



In a string theory,

one has to include the winding modes of strings around $\mathcal{A}$



These turn the theory on the defects supported on D ,

to a three dimensional quiver gauge theory on

$$\mathrm{D} \times S^1$$

where the S^1 is the dual of the circle in $\mathcal{A}$

The same T-duality that makes the D-branes

three dimensional turns them into monopoles on

$$\mathbb{R} \times \mathbb{C}$$

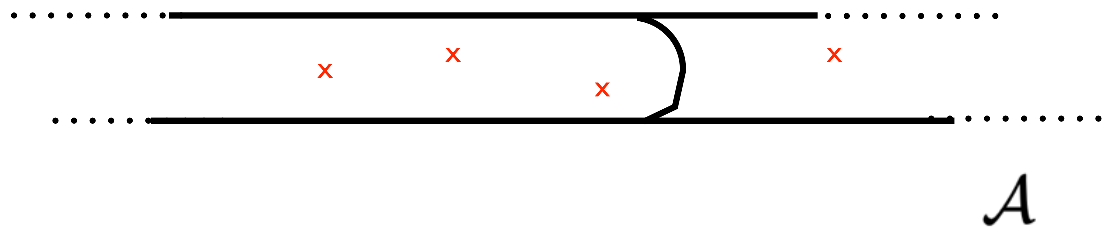
of the T-dual six dimensional (1,1) string

which is a gauge theory.

The choice of vertex operators in

$$\langle\lambda|\ \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\ |\lambda'\rangle$$

is the choice of D-branes of the little string.



The choice of the Verma module state $\langle\lambda|$

is the choice of moduli of little string theory,

i.e. they are the expectation values of dynamical fields.

One can study the three dimensional theory on

$$\mathrm{D} \times S^1$$

which comes from little string theory,

in much the same way

as we did the two dimensional theory.

The fact that the string scale is finite,

leads to a deformation of the structures

we had found, in particular, it breaks conformal invariance.

Rather than getting conformal blocks

and Knizhnik-Zamolodchikov equation,

from partition functions of the 3d theory on

$$\mathrm{D} \times S^1$$

one obtains their deformation

corresponding to replacing

$$\widehat{{}^L\mathfrak{g}} \longrightarrow U_\hbar(\widehat{{}^L\mathfrak{g}})$$

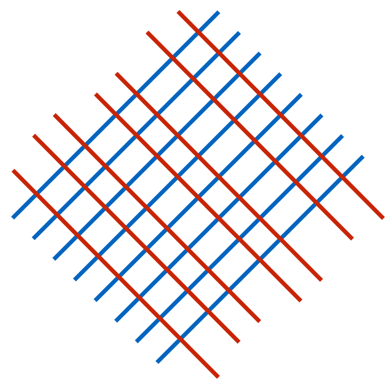
affine Lie algebra | quantum affine algebra

Pursuing our story further,

rather than discovering knot invariants

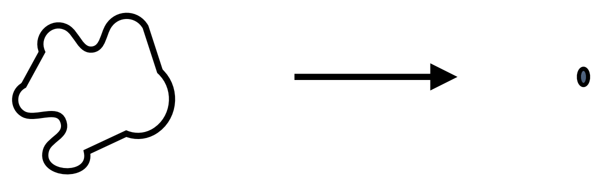
we would discover integrable lattice models,

those of, in some sense, very general kind.



This story is developed in the work with Andrei.

The six dimensional (2,0) string theory has a point particle limit



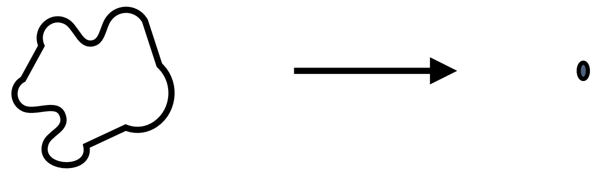
in which it becomes the six dimensional conformal field theory

of type $\mathfrak{g}$

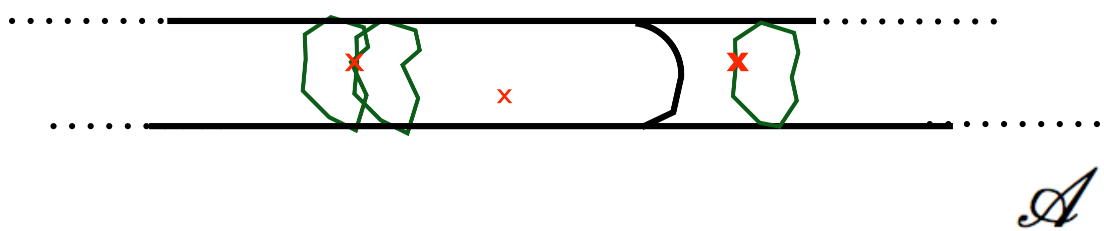
This limit coincides with the conformal limit of the quantum affine algebra

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

In the point particle limit,



the winding modes that made the theory
on the defects three dimensional, instead of two,
become infinitely heavy.



As a result, in the conformal limit, the theory on the defects
becomes a two dimensional theory on D

It is surprising, but by now well understood
that there are different two dimensional limits
a three dimensional gauge theory can have.

The point particle limit of little string theory
specifies which two dimensional limit
of the three dimensional gauge theory on a circle we need to take.

The resulting theory is not a gauge theory,
but it has the two other descriptions,
I described earlier in the talk,
related by two-dimensional mirror symmetry.

There is a third description,
due to Witten.

It describes the same physics,
just from the bulk perspective.

Compactified on a very small circle,

the six dimensional $\mathfrak{g}$ -type (2,0) conformal theory

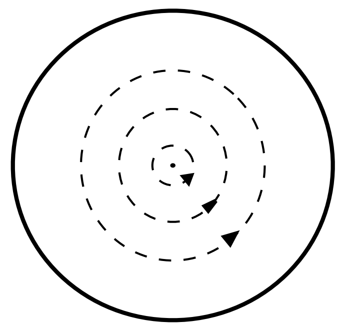
with no classical description,

becomes a $\mathfrak{g}$ -type gauge theory

in one dimension less.

To get a good 5d gauge theory description of the problem,

the circle one shrinks corresponds to S^1 in



so from a six dimensional theory on

$$M_6 \approx \mathcal{A} \times D \times \mathbb{C}$$

one gets a five-dimensional gauge theory on a manifold with a boundary

The five dimensional gauge theory is supported on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D} \quad \text{where} \quad \widetilde{M}_3 = \mathscr{A} \times \mathbb{R}_{\geq 0}$$

It has gauge group

$$G$$

which is the adjoint form of a Lie group with lie algebra $\mathfrak{g}$.

Our two dimensional defects become monopoles

of the 5d gauge theory on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D}$$

supported on D and at points on,

$$\widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0} \; ,$$

along its boundary.

Witten shows that the five dimensional theory on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D}$$

can be viewed as a gauged

Landau-Ginzburg model on D with potential

$$\mathcal{W}_{\mathrm{CS}} = \int_{\widetilde{M}_3} \mathrm{Tr}(A \wedge dA + A \wedge A \wedge A)$$

on an infinite dimensional target space $\mathcal{Y}_{\mathrm{CS}}$

corresponding to $\mathfrak{g}_{\mathbb{C}}$ connections on $\widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0}$

with suitable boundary conditions (depending on the knots).

To obtain knot homology groups in this approach,
one ends up counting solutions to
certain five dimensional equations.

The equations arise in
constructing the Floer cohomology groups
of the five dimensional Landau-Ginzburg theory.

Thus, we end up with three different approaches
to the knot categorification problem,
all of which have the same
six dimensional origin.

They all describe the same physics
starting in six dimensions.

The two geometric approaches,
describe the physics from perspective of the defects that introduce knots
in the theory.

The approach based on the 5d gauge theory,
describes it from perspective of the bulk.

In general,
theories on defects
capture only the local physics of the defect.

In this case,
they capture all of the relevant physics,
due to a version of supersymmetric localization:
in the absence of defects,
the bulk theory is trivial.