Please note: Questions 4 and 6 (marked with asterisks) should be handed in as part of **Assignment 2**, *due by* **noon on Friday 17 September**.

1. Let $\{Z_t\}$ be a stationary process with mean zero and variance σ_Z^2, and let a and b be constants. If
$$Y_t = a + bt + s_t + Z_t,$$
where s_t is a non-random seasonal component with period 12, show that $D(1 - B^{12})Y_t$ is stationary and express its autocovariance function in terms of the autocovariance function of $\{Z_t\}$.

   Reminder: D is the first difference operator, and B is the backward shift operator, both defined in lectures.

2. Show that the autocorrelation function for the process given by
$$Y_t = \sum_{j=0}^{m} Z_{t-j}/(m+1)$$
where Z_t is white noise, is
$$\rho_k = \begin{cases} (m+1-k)/(m+1) & k = 0, 1, 2, \ldots, m \\ 0 & \text{otherwise} \end{cases}$$

3. Let $\{Y_t\}$ $t = 1, 2, \ldots, n$ be a stationary time series.
   (a) Show that $\bar{Y}$ is an unbiased estimator of the mean, μ, of the series.
   (b) Let ρ_k be the *autocorrelation function* of Y_t. Show that
$$\mathrm{Var}\left[\bar{Y}\right] = \frac{\sigma^2}{n}\left\{1 + \frac{2}{n}\sum_{k=1}^{n-1}(n-k)\rho_k\right\}$$
   i.e. the variance of the mean depends on the autocorrelation structure.

4. * *A first order moving average process or MA(1) process*:

   Let $Y_t = Z_t + \beta Z_{t-1}$, for $t = 0, 1, \ldots$, where $\{Z_t\}$ is a white noise sequence with variance σ_Z^2 and β is a constant. Find the mean, variance and autocovariance function of $\{Y_t\}$.

5. *More general moving average processes*: Suppose that $\{Z_t\}$ is a white noise sequence with mean zero and variance σ_Z^2. Then a random process $\{Y_t\}$ is said to be a *moving average process of order* q, written MA(q), if

$$Y_t = Z_t + \beta_1 Z_{t-1} + \ldots + \beta_q Z_{t-q}$$

where $\beta_1, \ldots, \beta_q$ are constants.

(a) Find $E(Y_t)$ and $\mathrm{Var}\,[Y_t]$.

(b) Show that the autocovariance function of $\{Y_t\}$ is

$$\gamma_k = \sigma_Z^2 \sum_{i=0}^{q-k} \beta_i \beta_{i+k} \qquad \text{for} \quad k = 1, \ldots, q$$

and zero for $k > q$, where k is the lag.

6. * Find the general linear process representation of the AR(1) process

$$Y_t = 0.3 Y_{t-1} + Z_t$$

where Z_t is white noise. Is $\{Y_t\}$ is a stationary process? Justify your answer.

7. *We will prove the following theorem used in lectures if we have time:* Let $\omega_j = 2\pi j/n$ for some positive integer j less than $n/2$. Let $c_t = \cos(\omega t)$ and $s_t = \sin(\omega t)$. Then

$$\begin{aligned} \sum_{t=1}^{n} c_t &= \sum_{t=1}^{n} s_t = \sum_{t=1}^{n} c_t s_t = 0, \\ \sum_{t=1}^{n} c_t^2 &= \sum_{t=1}^{n} s_t^2 = n/2 \end{aligned}$$

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