3 Theory of stationary random processes

3.1 Linear filters and the General linear process

A filter is a transformation of one random sequence $\{U_t\}$ into another, $\{Y_t\}$.

- A **linear filter** is a transformation of the form

$$Y_t = \sum_{j=-\infty}^{\infty} a_j U_{t-j}.$$

  If $\{U_t\}$ is stationary and a finite number of the a_j are nonzero, then $\{Y_t\}$ is also stationary. When infinitely many a_j are nonzero, $\{Y_t\}$ may or may not be stationary, depending on the precise form of the a_j.

- **General linear process**: $Y_t = \sum_{j=0}^{\infty} a_j Z_{t-j}$ where Z_t is white noise and the a_j are constants.

- *Mean*: $\mathrm{E}[Y_t] = 0$

- *Variance*: $\mathrm{Var}[Y_t] = \sigma^2 \sum_{j=0}^{\infty} a_j^2$

- *Autocovariance*: $\mathrm{Cov}[Y_s, Y_t] = \sigma^2 \sum_{i=0}^{\infty} a_i a_{t-s+i}$

- *Stationary*: mean and variance are constant provided $\sum_{i=0}^{\infty} a_i^2 < \infty$. The autocovariance depends on $k = t - s$, again if the sum is finite.

- A necessary and sufficient condition for a general linear process to be stationary is

$$\sum_{i=0}^{\infty} a_i^2 < \infty$$

In practice, a general linear process is a useful model only when its coefficients a_j are expressible in terms of a finite number of parameters, which we can then hope to estimate from a set of data.

A very rich class of models which satisfies this requirement is the class of *autoregressive moving average (ARMA) processes*.

3.2 ARMA processes

$\{Z_t\}$ white noise is the basic building block of these processes.

- Moving Average, $Y_t \sim \text{MA}(q)$,

$$Y_t = Z_t + \sum_{j=1}^{q} \beta_j Z_{t-j}$$

- Autoregressive, $Y_t \sim \text{AR}(p)$,

$$Y_t = \sum_{i=1}^{p} \alpha_i Y_{t-i} + Z_t$$

- ARMA, $Y_t \sim \text{ARMA}(p, q)$

$$Y_t = \sum_{i=1}^{p} \alpha_i Y_{t-i} + Z_t + \sum_{j=1}^{q} \beta_j Z_{t-j}$$

- The above assume a zero mean, but can consider

$$Y_t - \mu = \sum_{i=1}^{p} \alpha_i (Y_{t-i} - \mu) + Z_t + \sum_{j=1}^{q} \beta_j Z_{t-j}$$

3.3 Backward Shift Operator

The above notation is relatively cumbersome, and it gets worse! We use the *backward shift operator* notation to simplify things.

- Recall: $BY_t = Y_{t-1}$, $B^jY_t = Y_{t-j}$, and $1Y_t = Y_t$.
- AR(p) may be written

$$(1 - \alpha_1 B - \cdots - \alpha_p B^p)Y_t = Z_t$$

- and the MA(q) may be written

$$Y_t = (1 + \beta_1 B + \cdots + \beta_q B^q)Z_t$$

- Combining for ARMA(p, q) model:

$$\phi(B)Y_t = \theta(B)Z_t,$$

$$\phi(B) = 1 - \sum_{i=1}^{p} \alpha_i B^i, \quad \theta(B) = 1 + \sum_{j=1}^{q} \beta_j B^j$$

- Note the side conditions

$$\phi(0) = \theta(0) = 1$$

These eliminate redundancy in the specification of $\{Y_t\}$.

- $$\phi(B)Y_t = \theta(B)Z_t$$

assumes no factors in common.

- Operations involving B are assumed to follow algebraic lines, so, for example,

$$\begin{aligned}\left(1 - \frac{B}{2}\right)^{-1} Y_t &= \left\{\sum_{j=0}^{\infty} \left(\frac{B}{2}\right)^j\right\} Y_t \\ &= \sum_{j=0}^{\infty} 2^{-j} B^j Y_t \\ &= \sum_{j=0}^{\infty} 2^{-j} Y_{t-j}\end{aligned}$$

3.4 Second-order properties of Moving Average process, MA(q)

Now, we know

$$Y_t = \theta(B)Z_t$$

where

$$\theta(B) = \sum_{j=0}^{q} \beta_j B^j$$

with $\beta_0 = 1$ and $\theta(0) = 1$.

- **Autocovariance function**:

$$\gamma_k = \begin{cases} \sigma^2 \sum_{i=0}^{q-k} \beta_{i+k}\beta_i & k = 0, 1, 2, \ldots, q \\ 0 & k > q \end{cases}$$

- **Autocorrelations**: for $k = 0, 1, 2, \ldots, q$

$$\rho_k = \frac{\sum_{i=0}^{q-k} \beta_{i+k}\beta_i}{\sum_{i=0}^{q} \beta_i^2}$$

- The ‘cut-off’ in γ_k after $k = q$ is a characteristic of the MA(q) process.

- **Stationarity**: for finite β_j, always stationary.

- **Invertibility**: Consider model MA(1):
  $Y_t = Z_t + \beta Z_{t-1}$

  - The only non-zero autocorrelation is
    $\rho_1 = \frac{\beta}{1+\beta^2}$

  - However if we replace β by $1/\beta$, ρ_1 doesn't change. Thus the model $Y_t = Z_t + \frac{1}{\beta} Z_{t-1}$ has the same second order properties as the model above. To resolve the ambiguity, write

$$Y_t = (1 + \beta B)Z_t$$

    and *invert* to give

$$Z_t = (1 + \beta B)^{-1} Y_t = \sum_{j=0}^{\infty} (-1)^j \beta^j Y_{t-j},$$

    expressing $\{Z_t\}$ as a linear filter of $\{Y_t\}$.

  - Invertibility: influence of present and past Y_t on Z_t vanishes as lag increases if and only if $-1 < \beta < 1$.

  - In general, invertibility means the roots of $\theta(u) = 0$ must be greater than 1 in absolute value.

Invertibility ensures a unique MA process for a given acf.

Comment: MA processes have been used in many areas, particularly econometrics where, for example, economic indicators are affected by a variety of 'random effects' such as strikes, terrorist attacks, government decisions, shortages of raw materials, etc. Such events will not only have an immediate effect, but may also affect economic indicators to a lesser extent in several subsequent periods. Thus it is at least plausible that an MA process may be an appropriate model.

Example: See handout on simulated MA(1) processes:
$\beta = -0.9, -0.5, 0.5, 0.9$, $n = 200$.

We will discuss the behaviour of these series in the lectures.

3.5 Autoregressive process, AR(p)

Recall: Y_t is expressed as a linear combination of the past Y_s and a white noise term Z_t. Note also that it expresses the white noise sequence $\{Z_t\}$ as a linear filter of $\{Y_t\}$.

We write

$$\phi(B)Y_t = Z_t$$

where

$$\phi(B) = 1 - \sum_{j=1}^{p} \alpha_j B^j$$

and $\phi(0) = 1$.

- **Stationary** if and only if all the roots of the equation $\phi(u) = 0$ have modulus greater than one (this is a theorem; for the proof, see handout and Diggle, p.76–77).

- **Example**: AR(1).

$$Y_t = \alpha Y_{t-1} + Z_t.$$

Thus,

$$\begin{aligned} Y_t - \alpha Y_{t-1} &= Z_t \\ \text{i.e.} \quad (1-\alpha B)Y_t &= Z_t, \end{aligned}$$

where $(1 - \alpha u) = \phi(u)$. This has root $1/\alpha$. Provided $|\alpha| < 1$, then $|1/\alpha| > 1$ as required for stationarity.

Recall earlier work with this example where we assumed this result.

- **Autocovariance function for AR(p) process**: in general, need to solve the **Yule-Walker** equations

$$\rho_k = \sum_{j=1}^{p} \alpha_j \rho_{k-j}, \quad k = 1, 2, \ldots$$

To see this, consider

$$Y_t = \sum_{j=1}^{p} \alpha_j Y_{t-j} + Z_t.$$

Multiply both sides by Y_{t-k} to give

$$Y_t Y_{t-k} = \sum_{j=1}^{p} \alpha_j Y_{t-j} Y_{t-k} + Z_t Y_{t-k}.$$

Taking expectations of both sides leads to

$$\gamma_k = E(Y_t Y_{t-k}) = \sum_{j=1}^{p} \alpha_j \gamma_{k-j}$$

(check this as an exercise).

Finally, dividing both sides by $\gamma_0 = \text{var}(Y_t)$ gives the autocorrelation coefficients

$$\rho_k = \sum_{j=1}^{p} \alpha_j \rho_{k-j}, \quad k = 1, 2, \ldots.$$

This gives a way of calculating the ρ_k from the α_j: can solve by successive substitution, or solve as a system of linear difference equations.

Difference Equations: pth order, constant coefficients

$$\lambda_{k+p} + a_1\lambda_{k+p-1} + \cdots + a_p\lambda_k = b_k$$

- $a_1, a_2, \ldots, a_p$ are known constants, b_k an arbitrary function of k.
- *Homogeneous equation*: $b_k = 0$
  - *auxiliary equation*: $\lambda_k = \lambda^k$ is a solution iff $\lambda^p + a_1\lambda^{p-1} + \cdots + a_p = 0$
  - distinct roots $r_1, r_2, \ldots, r_p$, the solution is $\lambda_k = c_1 r_1^k + \cdots + c_p r_p^k$
  - some equal roots, for example $r_1 = r_2 = \cdots = r_m = r$ for some m, $\lambda_k = (c_1 + c_2 k + \cdots c_m k^{m-1})r^k + c_{m+1}r_{m+1}^k + \cdots + c_p r_p^k$
  - the constants c_i are determined by initial conditions.
- *Solution to general equation*: general solution to homogeneous equation plus any particular solution to the general equation.

Thus we look for solutions to the Yule-Walker equations of the form

$$\rho_k = \lambda^k$$

The *auxiliary equation* is

$$\lambda^p - \alpha_1\lambda^{p-1} - \cdots - \alpha_p = 0$$

In fact, the general solution for distinct roots $\lambda_1, \ldots, \lambda_p$, has the form

$$\rho_k = c_1\lambda_1^k + \cdots + c_p\lambda_p^k$$

[Note that the Yule-Walker equations can be used to infer values of α_j corresponding to an observed set of sample autocorrelation coefficients: this suggests a method of parameter estimation for autoregressive processes.]

Example: AR(2). The general form of this process is

$$Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + Z_t.$$

This model generates a rich variety of second-order properties depending on the numerical values of the parameters α_1 and α_2.

Write model as

$$Y_t - \alpha_1 Y_{t-1} - \alpha_2 Y_{t-1} = Z_t \quad (*)$$

Find ρ_k.

The Yule-Walker equations are

$$\rho_k = \alpha_1 \rho_{k-1} + \alpha_2 \rho_{k-2}, \quad k > 0$$

We look for solutions of the form $\rho_k = \lambda^k$.

Substitute into $(*)$ to give

$$\lambda^k = \alpha_1 \lambda^{k-1} + \alpha_2 \lambda^{k-2},$$

i.e.,

$$\lambda^2 - \alpha_1 \lambda - \alpha_2 = 0,$$

the auxiliary equation.

We know that solutions to the auxiliary equation are solutions to $(*)$ as well.

This quadratic equation has two roots:

$$\frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2}$$

Call these

$$\lambda_1 = \frac{1}{2}\left(\alpha_1 + \sqrt{\alpha_1^2 + 4\alpha_2}\right),$$

$$\lambda_2 = \frac{1}{2}\left(\alpha_1 - \sqrt{\alpha_1^2 + 4\alpha_2}\right),$$

Now, the nature of the solution depends on whether the roots are

- real and distinct, i.e., $\alpha_1^2 + 4\alpha_2 > 0$
- real and coincident, i.e., $\alpha_1^2 + 4\alpha_2 = 0$, or
- complex, i.e., $\alpha_1^2 + 4\alpha_2 < 0$.

In the lectures, we will obtain the general solution for each case and consider some illustrative examples.

3.6 ARMA(p,q) processes

We write

$$\phi(B)Y_t = \theta(B)Z_t$$

where

$$\phi(B) = 1 - \sum_{j=1}^{p} \alpha_j B^j, \quad \theta(B) = 1 + \sum_{j=1}^{q} \beta_j B^j,$$

and $\phi(0) = 1$, $\theta(0) = 1$.

- **Autocorrelation function** is found in specific cases by expressing Y_t as a general linear process, that is

  $$Y_t = \{\phi(B)\}^{-1}\theta(B)Z_t = \left(\sum_{j=0}^{\infty} a_j B^j\right) Z_t$$

  where the coefficients a_j are determined by a formal power series expansion of $\{\phi(B)\}^{-1}$.

- **Stationarity** is determined by the AR part, that is, if the roots of $\phi(u) = 0$ are greater than one in absolute value.

- **Invertibility**: the ARMA(p,q) process is *invertible* if the roots of the polynomial equation $\theta(u) = 0$ are all greater than one in absolute value.

Example: the ARMA(1,1) process.

See also separate handout of simulated ARMA processes.

ARMA processes are valuable in time series analysis owing to their ability to approximate a wide range of second-order behaviour using only a small number of parameters.

Additional motivation is their central role in Box-Jenkins forecasting, and their extension to ARIMA processes. Further comments are made at the end of Chapter 4.