[No. of pages: 7]
[No. of questions: 6]
[Total marks: 120]

**THE UNIVERSITY OF ADELAIDE**

**SCHOOL OF APPLIED MATHEMATICS**

**LEVEL IV STATISTICS HONOURS**

# TIME SERIES

**NOVEMBER 2003**

**Time: 3 hours**

[In addition, candidates are allowed ten minutes before the exam begins, to read the paper.]

**Answer all questions.**
**Each question is worth 20 marks.**
**Pages 4 to 6 attached contain plots referred to in the written questions.**
**A Formula Sheet is attached on Page 7.**
**Calculators are permitted for simple numerical calculations.**

1. (a) Explain briefly why time series require dedicated statistical methods for analysis.

   (b) Let $\{Y(t)\}$ be a random function.

   (i) Define the trend in mean and the autocovariance function of $\{Y(t)\}$.

   (ii) Define what is meant by the statement that $\{Y(t)\}$ is *second-order* or *weakly stationary.*

   (iii) What is the role of stationarity in time series?

   [*Note: hereafter in this examination paper, we assume a stationary process is second-order stationary.*]

   (c) Let $\{Y_t\}$ be the simple random walk defined by

$$Y_t = Y_{t-1} + Z_t$$

   where $\{Z_t\}$ is a white noise sequence with mean zero and variance $\sigma_Z^2$, $Y_0 = 0$ and $t = 1, 2 \ldots.$

   (i) Show that $\text{var}(Y_t) = t\sigma_Z^2$.

   (ii) Explain briefly why the result in (i) demonstrates that the simple random walk is non-stationary.

2. (a) A stationary random process $\{Y_t\}$ has mean $\mu$ and autocovariance function $\gamma_Y(k)$. Show that the new process $\{D_t\}$, where $D_t = Y_t - Y_{t-1}$, is stationary and find its autocovariance function in terms of $\gamma_Y(k)$.

   (b) State two purposes of smoothing an observed time series.

   (c) Suppose $\{Z_t\}$ is a white noise sequence with variance $\sigma^2$, and that

$$S_t = (Z_{t-1} + Z_t + Z_{t+1})/3$$

   is the simple three-point moving average. Find the autocovariance function of the residual sequence $\{R_t\}$, where $R_t = Z_t - S_t$.

3. (a) Define the *sample autocorrelation coefficients*, $r_k$, for lags $k = 0, 1, \ldots.$

   (b) What is the *correlogram*? How is the correlogram used to assess whether a series is consistent with white noise?

   (c) Comment on the implications of the result in Question 2(c) for examining the correlogram of a residuals series after removing trend using a simple three-point moving average.

   (d) The top picture in Figure 1 shows a time series of 468 monthly atmospheric carbon dioxide concentrations (in parts per million) observed at Mauna Loa from 1959 to 1997. The bottom picture in Figure 1 shows the associated correlogram. Figure 2 gives the periodogram of the series after the mean and linear trend have been removed.

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(i) Describe the features of the $CO_2$ time series. Is the series non-stationary? Justify your answer.

(ii) Interpret and explain the information given about the series in the periodogram.

(e) Show that the periodogram ordinates and the sample autocovariance function are related by

$$I(\omega) = g_0 + 2\sum_{k=1}^{n-1} g_k \cos(k\omega)$$

where $g_0$ and $g_k$ are the sample variance and autocovariances, respectively.

[*You may assume the result* $\sum_{t=1}^{n} \cos(\omega t) = \sum_{t=1}^{n} \sin(\omega t) = 0$.]

4. A first-order moving average process, MA(1), is defined by

$$Y_t = Z_t + \beta Z_{t-1}$$

where $|\beta| < 1$, and $\{Z_t\}$ is white noise with mean zero and variance $\sigma_Z^2$.

(a) Find the mean, variance and autocorrelation function of $\{Y_t\}$.

(b) This process is invertible. Explain what is meant by *invertibility*.

(c) Show that the spectrum for the MA(1) process is

$$f(\omega) = \sigma_Z^2\{1 + \beta^2 + 2\beta\cos(\omega)\}$$

(d) Figure 3 (top plot) gives a realisation of $n = 200$ simulated observations from a first-order moving average process $Y_t = Z_t + 0.9Z_{t-1}$. Figure 3 also shows the correlogram and periodogram for the observed series, the theoretical autocorrelation function, and the spectrum.

Relate the form of the observed series to the theoretical properties of the underlying process.

5. (a) Find the autocorrelation function for the AR(2) process

$$Y_t = Y_{t-1} + cY_{t-2} + Z_t$$

where $c = -3/16$.

(b) An ARMA$(1, 1)$ process is defined by

$$Y_t = 0.5Y_{t-1} + Z_t - 0.3Z_{t-1}$$

where $\{Z_t\}$ is white noise with zero mean and variance $\sigma^2$.

(i) Express the model using the backward shift operator $B$. Hence, or otherwise, determine whether the process is stationary and/or invertible.

(ii) Express $Y_t$ as a general linear process.

6. (a) Show that the autocorrelation function of the ARMA(1,1) process

$$Y_t = \alpha Y_{t-1} + Z_t + \beta Z_{t-1}$$

is given by

$$\begin{aligned}\rho_1 &= \frac{(1+\alpha\beta)(\alpha+\beta)}{(1+\beta^2+2\alpha\beta)} \\ \rho_k &= \alpha\rho_{k-1} \quad \text{for} \quad k = 2, 3, \ldots\end{aligned}$$

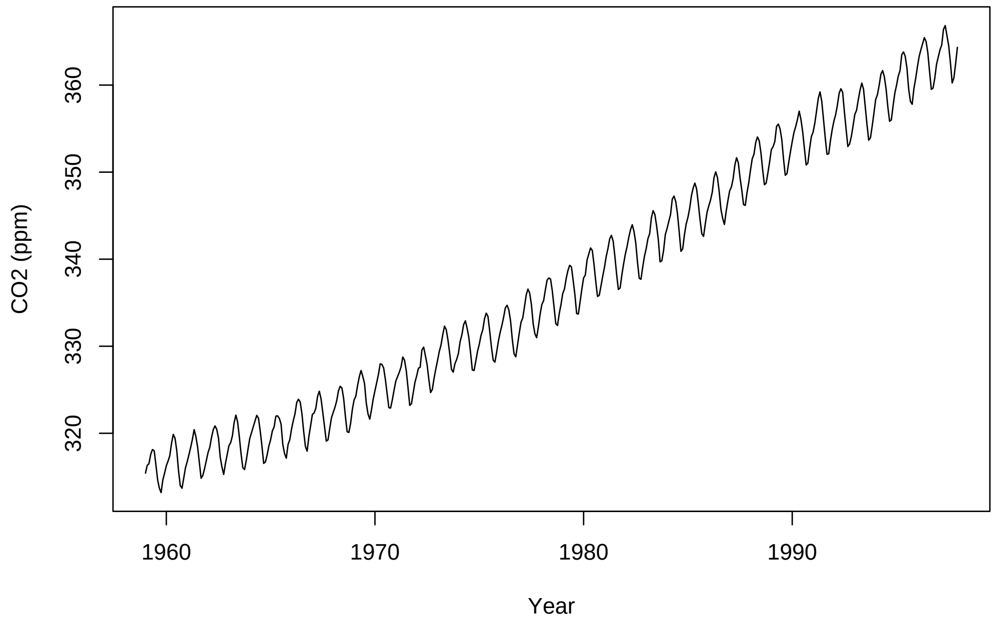
(b) Consider the ARIMA process defined by

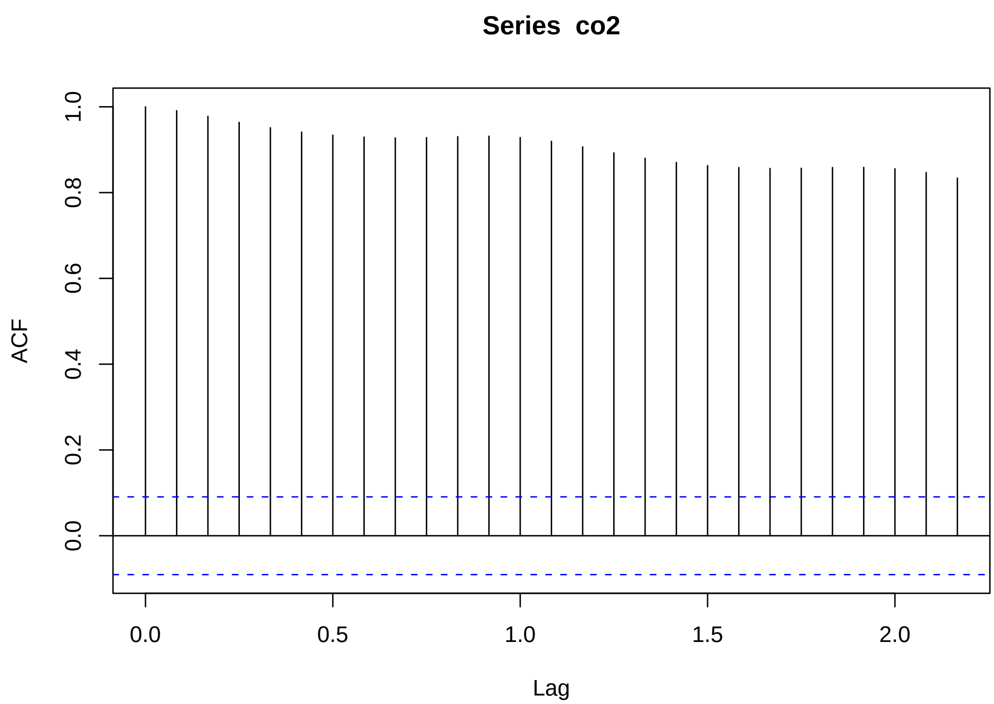
$$(1 - 0.7B)(1 - B)Y_t = Z_t$$

(i) Identify $p, d, q$ in this process.

(ii) Is the process stationary? Justify your answer.

# Figure 1: Mauna Loa atmospheric $CO_2$ concentration

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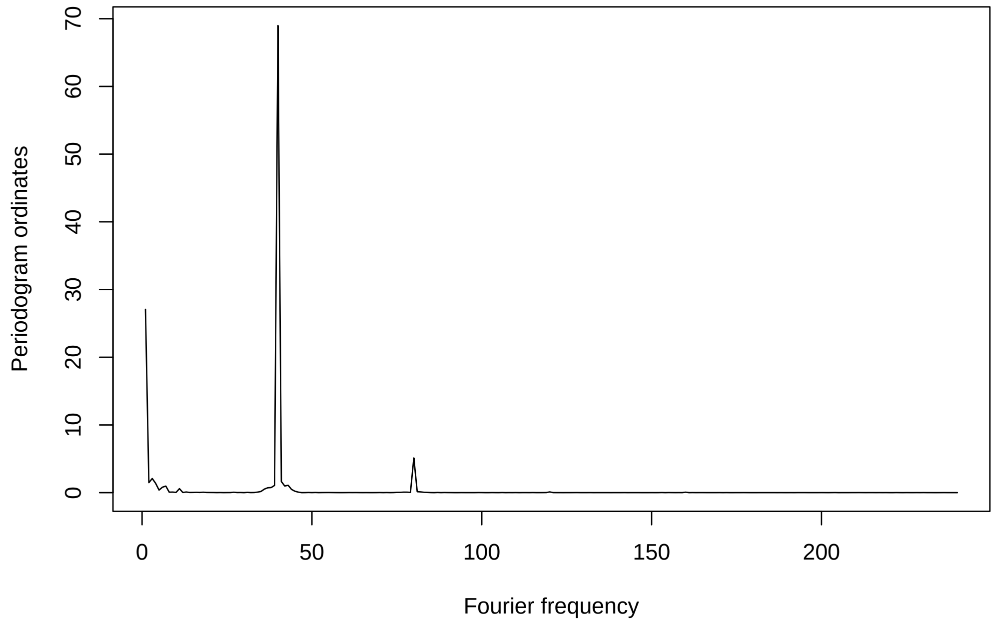
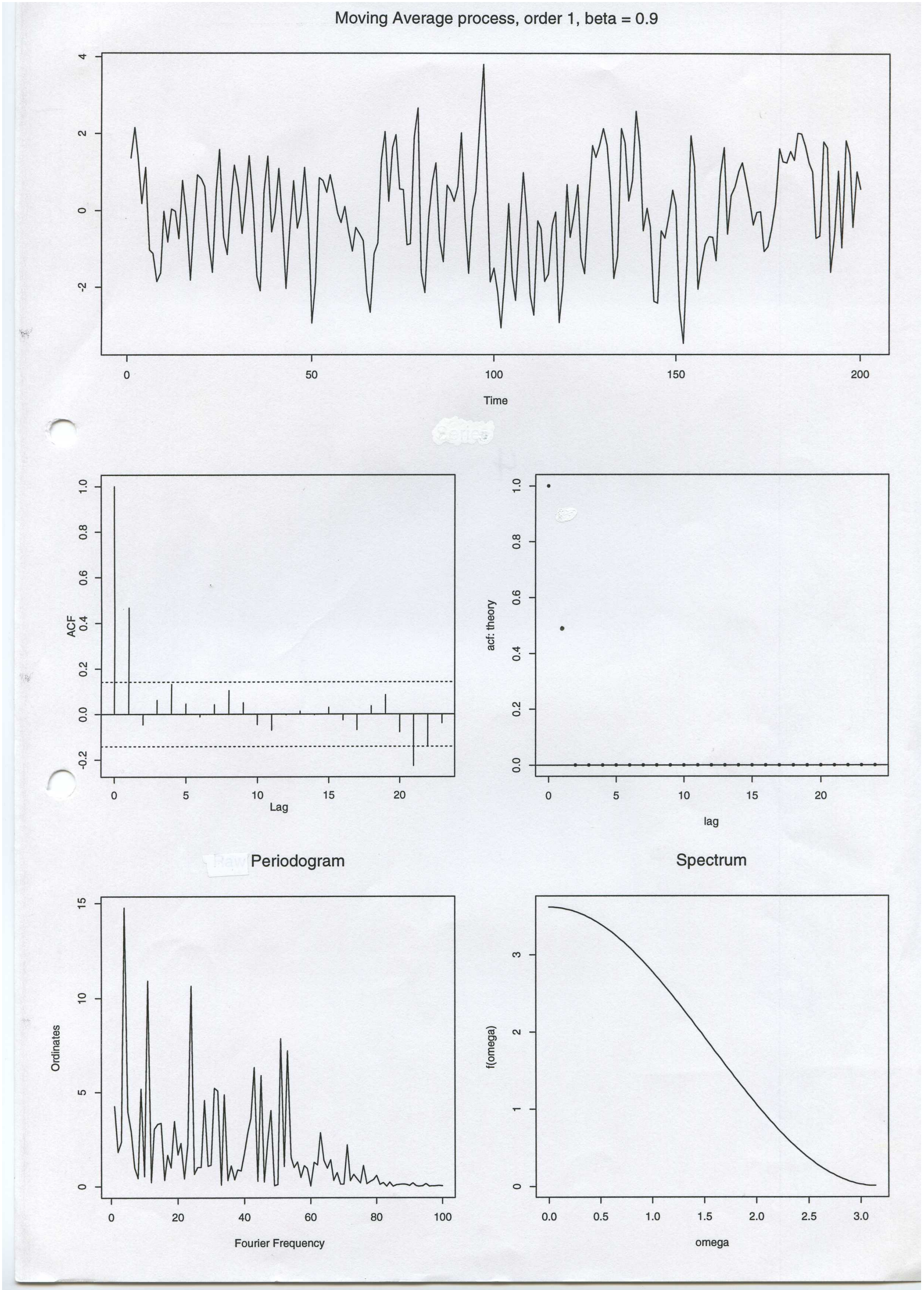
## Figure 2: Periodogram of series $CO_2$



## Figure 3: MA(1) process $Y_t = Z_t + 0.9Z_{t-1}$

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# Formula Sheet for Time Series

**Moving averages**: A moving average of order $2p+1$ of a time series $\{y_t : t = 1, 2, \ldots, n\}$ is a time series defined by

$$s_t = \sum_{j=-p}^{p} w_j y_{t+j}, \quad t = p+1, \ldots, n-p$$

**Sines and cosines**:

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

**Periodogram ordinates**:

$$I(\omega) = \frac{1}{n}\left\{\left(\sum_{t=1}^{n} y_t \cos(\omega t)\right)^2 + \left(\sum_{t=1}^{n} y_t \sin(\omega t)\right)^2\right\}$$

where $0 < \omega \leq \pi$.

**Yule-Walker equations for AR($p$) process**:

$$\rho_k = \sum_{j=1}^{p} \alpha_j \rho_{k-j}, \quad k = 1, 2, \ldots$$

**ARMA($p, q$) process**:

$$Y_t - \alpha_1 Y_{t-1} - \cdots - \alpha_p Y_{t-p} = Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}$$

**Spectrum and normalized spectrum**:

$$f(\omega) = \gamma_0 + 2\sum_{k=1}^{\infty} \gamma_k \cos(k\omega)$$

$$f^*(\omega) = 1 + 2\sum_{k=1}^{\infty} \rho_k \cos(k\omega)$$