Please note: Questions 1 and 5 (marked with asterisks) should be handed in as part of Assignment 1, due by 4pm on Friday 29 August.

1. * Covariances of linear combinations of random variables. For constants $a$, $b$, $c$ and $d$, show that
   
   (a) $\text{Cov}[a + X, Y] = \text{Cov}[X, Y]$
   (b) $\text{Cov}[aX, bY] = ab\text{Cov}[X, Y]$
   (c) $\text{Cov}[X, Y + Z] = \text{Cov}[X, Y] + \text{Cov}[X, Z]$
   (d) Hence show
      
      $\text{Cov}[aW + bX, cY + dZ] = ac\text{Cov}[W, Y] + bc\text{Cov}[X, Y] + ad\text{Cov}[W, Z] + bd\text{Cov}[X, Z]$

2. Suppose $X$ and $Y$ are random variables with means 3 and 5 respectively. If $X$ and $Y$ have variances 1 and 2 respectively and covariance 1, find
   
   (a) the correlation of $X$ and $Y$
   (b) $\text{Var}[2X + 3Y]$ and $\text{Var}[X - Y]$
   (c) $\text{Cov}[2X + 3Y, X - Y]$
   (d) the correlation of $2X + 3Y$ with $X - Y$

3. Assume random variables $X$ and $Y$ have variances $\sigma_X^2$ and $\sigma_Y^2$, covariance $\sigma_{XY}$ and correlation $\rho$. Show that
   
   $\text{Var}\left[\frac{X}{\sigma_X} \pm \frac{Y}{\sigma_Y}\right] = 2(1 \pm \rho)$
   
   and hence show $-1 \leq \rho \leq 1$.

4. Let $\{Z_t\}$ be a sequence of independent normal random variables, each with mean zero and variance $\sigma^2$, and let $a$, $b$ and $c$ be constants. Which, if any, of the following processes are stationary? For each stationary process specify the mean and autocovariance function.
   
   (a) $Y_t = a + bZ_t + cZ_{t-2}$
   (b) $Y_t = Z_1 \cos(ct) + Z_2 \sin(ct)$
   (c) $Y_t = Z_t \cos(ct) + Z_{t-1} \sin(ct)$
   (d) $Y_t = Z_t Z_{t-1}$

5. * Let $\{Z_t\}$ be a stationary process with mean zero and let $a$ and $b$ be constants. If
   
   $Y_t = a + bt + s_t + Z_t$,

   where $s_t$ is a non-random seasonal component with period 12, show that $D(1 - B^{12})Y_t$ is stationary and express its autocovariance function in terms of the autocovariance function of $\{Z_t\}$.

   *Reminder: $D$ is the first difference operator, and $B$ is the backward shift operator, both defined in lectures.*
6. Suppose we have a random sequence \( \{U_t\} \) which is white noise. If we smooth the sequence using a moving average then remove the smoothed trend, this induces spurious autocorrelation into the residual series, as you will show.

Define \( R_t = U_t - S_t \), where

\[
S_t = \frac{(U_{t-1} + U_t + U_{t+1})}{3}
\]

is the simple moving average of order 3. Find the autocovariance function of \( \{R_t\} \).

7. The first 10 sample autocorrelation coefficients of 400 ‘random’ numbers are \( r_1 = 0.02 \), \( r_2 = 0.05 \), \( r_3 = -0.09 \), \( r_4 = 0.08 \), \( r_5 = -0.02 \), \( r_6 = 0.00 \), \( r_7 = 0.12 \), \( r_8 = 0.06 \), \( r_9 = 0.02 \) and \( r_{10} = -0.08 \). Is there any evidence of non-randomness?

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13 August 2003