1. DISTRIBUTION THEORY

1.4 CDF transformation

Suppose $X$ is a continuous RV with CDF $F_X(x)$, which is increasing over the range of $X$. If $U = F_X(x)$, then $U \sim U(0,1)$.

Proof.

$$F_U(u) = P(U \leq u)$$

$$= P\{F_X(X) \leq u\}$$

$$= P\{X \leq F_X^{-1}(u)\}$$

$$= F_X\{F_X^{-1}(u)\}$$

$$= u, \quad \text{for } 0 < u < 1.$$  

This is simply the CDF of $U(0,1)$, so the result is proved. \qed

The converse also applies. If $U \sim U(0,1)$ and $F$ is any strictly increasing “on its range” CDF, then $X = F^{-1}(U)$ has CDF $F(x)$, i.e.,

$$F_X(x) = P(X \leq x) = P\{F^{-1}(U) \leq x\}$$

$$= P(U \leq F(x))$$

$$= F(x), \quad \text{as required.}$$

1.5 Non-monotonic transformations

Theorem 1.3.2 applies to monotonic (either strictly increasing or decreasing) transformations of a continuous RV.

In general, if $h(x)$ is not monotonic, then $h(x)$ may not even be continuous.

For example, integer part of $x$

if $h(x) = \lfloor x \rfloor$,

then possible values for $Y = h(X)$ are the integers

$\Rightarrow Y$ is discrete.
However, it can be shown that \( h(X) \) is continuous if \( X \) is continuous and \( h(x) \) is piecewise monotonic. 

1.6 Moments of transformed RVS

Suppose \( X \) is a RV and let \( Y = h(X) \).

If we want to find \( E(Y) \), we can proceed as follows:

1. Find the distribution of \( Y = h(X) \) using preceding methods.

\[
E(Y) = \begin{cases} 
\sum_y y p(y) & Y \text{ discrete} \\
\int_{-\infty}^{\infty} y f(y) \, dy & Y \text{ continuous}
\end{cases}
\]

(that is, forget \( X \) ever existed!)

Or use

**Theorem. 1.6.1**

If \( X \) is a RV of either discrete or continuous type and \( h(x) \) is any transformation (not necessarily monotonic), then \( E\{h(X)\} \) (provided it exists) is given by:

\[
E\{h(X)\} = \begin{cases} 
\sum_x h(x) p(x) & X \text{ discrete} \\
\int_{-\infty}^{\infty} h(x) f(x) \, dx & X \text{ continuous}
\end{cases}
\]

Proof.

Not examinable. \( \square \)

**Examples:**

1. CDF transformation

Suppose \( U \sim U(0,1) \). How can we transform \( U \) to get an \( \text{Exp}(\lambda) \) RV?
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Solution. Take \( X = F^{-1}(U) \), where \( F \) is the \( \text{Exp}(\lambda) \) CDF. Recall \( F(x) = 1 - e^{-\lambda x} \) for the \( \text{Exp}(\lambda) \) distribution.

To find \( F^{-1}(u) \), we solve for \( x \) in \( F(x) = u \), i.e.,

\[
\begin{align*}
    u &= 1 - e^{-\lambda x} \\
    \Rightarrow 1 - u &= e^{-\lambda x} \\
    \Rightarrow \ln(1 - u) &= -\lambda x \\
    \Rightarrow x &= -\frac{\ln(1 - u)}{\lambda}.
\end{align*}
\]

Hence if \( U \sim U(0, 1) \), it follows that \( X = -\frac{\ln(1 - U)}{\lambda} \sim \text{Exp}(\lambda) \).

Note: \( Y = -\frac{\ln U}{\lambda} \sim \text{Exp}(\lambda) \) [both \( 1 - U \) & \( U \) have \( U(0, 1) \) distribution].

This type of result is used to generate random numbers. That is, there are good methods for producing \( U(0, 1) \) (pseudo-random) numbers. To obtain \( \text{Exp}(\lambda) \) random numbers, we can just get \( U(0, 1) \) numbers and then calculate \( X = -\frac{\ln U}{\lambda} \).

2. Non-monotonic transformations

Suppose \( Z \sim N(0, 1) \) and let \( X = Z^2 \); \( h(Z) = Z^2 \) is not monotonic, so Theorem 1.3.2 does not apply. However we can proceed as follows:

\[
F_X(x) = P(X \leq x)
\]

\[
= P(Z^2 \leq x)
\]

\[
= P(-\sqrt{x} \leq Z \leq \sqrt{x})
\]

\[
= \Phi(\sqrt{x}) - \Phi(-\sqrt{x}),
\]

where

\[
\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt
\]

is the \( N(0, 1) \) CDF. That is,

\[
\Rightarrow f_X(x) = \frac{d}{dx} F_X(x) = \phi(\sqrt{x}) \left( \frac{1}{2} x^{-1/2} \right) - \phi(-\sqrt{x}) \left( -\frac{1}{2} x^{-1/2} \right),
\]

\[
\text{check! chain rule}
\]

Differentiate
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where $\phi(a) = \Phi'(a) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}$ is the $N(0, 1)$ PDF

$$= \frac{1}{2^\frac{1}{2}} \frac{1}{\sqrt{2\pi}} e^{-x/2} + \frac{1}{\sqrt{2\pi}} e^{-x/2}$$

$$= \frac{(1/2)^{1/2}}{\sqrt{\pi}} x^{1/2} e^{-1/2x},$$
which is the Gamma $\Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$ PDF.

On the other hand, the distribution of $Z^2$ is also called the $\chi^2_1$ distribution. We have proved that it is the same as the Gamma$(\frac{1}{2}, \frac{1}{2})$ distribution.

3. Moments of transformed RVS:

Example: if $U \sim U(0, 1)$ and $Y = -\log U / \lambda$ then $Y \sim \text{Exp}(\lambda) \Rightarrow f(y) = \lambda e^{-\lambda y}, \ y > 0$.

Can check

$$E(Y) = \int_0^\infty \lambda y e^{-\lambda y} dy$$

$$= \frac{1}{\lambda}.$$  

4. Based on Theorem 1.6.1:

If $U \sim U(0, 1)$ and $Y = -\log U / \lambda$, then according to Theorem 1.6.1,

$$E(Y) = \int_0^1 \frac{-\log u}{\lambda} (1) du$$

$$= -\frac{1}{\lambda} \int_0^1 \log u \ du = -\frac{1}{\lambda} \int_0^1 (u \log u - u) \ du$$

$$= -\frac{1}{\lambda} \left[ u \log u \right]_0^1 - \frac{1}{\lambda} \left[ u \right]_0^1$$

$$= -\frac{1}{\lambda} [0 - 1]$$

$$= \frac{1}{\lambda}, \ as \ required.$$
There are some important consequences of Theorem 1.6.1:

1. If \( E(X) = \mu \) and \( \text{Var}(X) = \sigma^2 \), and \( Y = aX + b \) for constants \( a, b \), then \( E(Y) = a\mu + b \) and \( \text{Var}(Y) = a^2\sigma^2 \).

   \textbf{Proof.} (Continuous case)

   \[
   E(Y) = E(aX + b) = \int_{-\infty}^{\infty} (ax + b)f(x) \, dx
   \]

   \[
   = a\int_{-\infty}^{\infty} xf(x) \, dx + b\int_{-\infty}^{\infty} f(x) \, dx
   \]

   \[
   = aE(X) + b
   \]

   \[
   = a\mu + b.
   \]

   \[
   \text{Var}(Y) = E \left[ (Y - E(Y))^2 \right]
   \]

   \[
   = E \left[ (aX + b - (a\mu + b))^2 \right]
   \]

   \[
   = E \left[ a^2(X - \mu)^2 \right]
   \]

   \[
   = a^2E \left( (X - \mu)^2 \right)
   \]

   \[
   = a^2 \text{Var}(X)
   \]

   \[
   = a^2\sigma^2.
   \]

2. If \( X \) is a RV and \( h(X) \) is any function, then the MGF of \( Y = h(X) \), provided it exists is,

\[
E \left[ e^{ty} \right] = M_Y(t) = \begin{cases} 
\sum_x e^{th(x)}p(x) & X \text{ discrete} \\
\int_{-\infty}^{\infty} e^{th(x)}f(x) \, dx & X \text{ continuous}
\end{cases}
\]
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This gives us another way to find the distribution of $Y = h(X)$.
i.e., Find $M_Y(t)$. If we recognise $M_Y(t)$, then by uniqueness we can conclude that
$Y$ has that distribution.

Examples

1. Suppose $X$ is continuous with CDF $F(x)$, and $F(a) = 0$, $F(b) = 1$; ($a$, $b$ can be
$\pm \infty$ respectively).
Let $U = F(X)$. Observe that

$$E\left[e^{tF(x)}\right] = M_U(t) = \int_a^b e^{tF(x)} f(x) \, dx$$

$$= \frac{1}{t} \left. e^{tF(x)} \right|_a^b = \frac{e^{tF(b)} - e^{tF(a)}}{t}$$

$$= \frac{e^t - 1}{t},$$

which is the $U(0,1)$ MGF.

2. Suppose $X \sim U(0,1)$, and let $Y = -\frac{\log X}{\lambda}$.

$$E\left[e^{tY}\right] = M_Y(t) = \int_0^1 e^{t(-\frac{\log x}{\lambda})} (1) \, dx = \int_0^1 e^{-\frac{t}{\lambda} \log x} \, dx$$

$$= \int_0^1 x^{-\frac{t}{\lambda}} \, dx$$

$$= \frac{1}{1 - t/\lambda} \left| \frac{1}{0} \right|$$

$$= \frac{1}{1 - t/\lambda}$$

$$= \frac{\lambda}{\lambda - t}, \quad t < \lambda$$

which is the MGF for $\text{Exp}(\lambda)$ distribution. Hence we can conclude that

$$Y = -\frac{\log X}{\lambda} \sim \text{Exp}(\lambda).$$

Again, recovered earlier result, here using

MGF.