ADELAIDE UNIVERSITY

DEPARTMENT OF STATISTICS

LEVEL II STATISTICS

(4107)  INTRODUCTION TO MATHEMATICAL STATISTICS

JUNE 2001

Time:  2 hours and 10 minutes

[It is recommended that you use the first 10 minutes of this time to carefully read the paper, before commencing your answers.]

Answer all questions.

Approximate marks for each question are indicated.

Please note that the questions are worth different numbers of marks.

Formulae sheets and a table of normal probabilities are attached.

Calculators are permitted for simple numerical calculations.
1. Consider a system with two electrical components. The first component has a probability of failure of 10%. If the first component fails, then the second also fails with probability 20%. If the first component works, then the second fails with probability 5%.

(a) Find the probability that at least one component works.
(b) Find the probability that exactly one component works.

[6 marks]

2. In a 1990 a man was sentenced to 16 years in jail for raping three women. The main evidence linking the accused to the attacks was the allegedly close match between the DNA genetic fingerprint of samples found at the scene of the crime and the accused. At the trial, the forensic scientist said that the match was so good, that only 1 in 3,000,000 people would match this. But in 1993, the Court of Appeal quashed the conviction declaring that the verdict was unsafe and that the forensic scientist and judge had fallen into a trap known as the ‘prosecutor’s fallacy’. The DNA evidence gives the probability of getting as good a DNA match from a randomly chosen (innocent) person in the population, i.e. $P(\text{match}|I)$, where $I$ denotes ‘innocent’. However the judge and forensic scientist confused this with the probability of the accused being innocent given so good a match, i.e. $P(I|\text{match})$.

The purpose of this question is to do the correct analysis.

(a) There was little other corroborating evidence, so assume that the prior probability of guilt was one in a million i.e. $1/(1,000,000)$. What is the prior odds of guilt?

(b) Assuming the probability of a match given someone is guilty is one, find the likelihood ratio, i.e.

$$\frac{P(\text{match}|G)}{P(\text{match}|I)},$$

where $G$ denotes ‘guilty’.

(c) Hence find the posterior odds of guilt.

Do you think your result corresponds to ‘guilty beyond reasonable doubt’? Justify your answer.

[10 marks]

3. (a) Define the expectation (or expected value) of a discrete random variable $Y$.

(b) The variance of a random variable $Y$ is $\sigma^2 = E[(Y - \mu_Y)^2]$. Show that this is equal to

$$E(Y^2) - \mu_Y^2,$$

where $\mu_Y$ is the (population) mean of $Y$.

(c) Let $Y$ be a Poisson random variable with rate $\mu$. Show that the moment generating function (mgf) of $Y$ is

$$m(t) = \exp\{\mu(e^t - 1)\}.$$ Use the mgf to find the mean and variance of $Y$.

[16 marks]
4. The mint produces 5c coins which have an average diameter of 12mm and a standard deviation of 0.2mm.
   (a) Use Tchebychev’s Inequality to find a lower bound for the proportion of coins that are expected to have diameters between 11.6 and 12.4mm.
   (b) Now suppose that the diameters are normally distributed. What is the probability that a randomly selected coin has a diameter inside the stated limits?
   (c) In a sample of 400, what number would you expect to be within the limits?

[10 marks]

5. The magnitude $T$ of earthquakes recorded in a region of North America can be modelled as having an exponential distribution with mean 2.4 as measured on the Richter scale.
   (a) Find the probability that an earthquake striking this region will exceed 5.0 on the Richter scale.
   (b) Find the probability that an earthquake striking this region will fall between 2.0 and 3.0 on the Richter scale.
   (c) Find the median earthquake magnitude.

[8 marks]

6. Consider the joint density function

$$f(x, y) = \exp(-x), \quad 0 \leq y \leq x < \infty,$$

and zero elsewhere.
   (a) Find and name the marginal density function of $X$.
   (b) Find and name the marginal density function of $Y$.
   (c) Are $X$ and $Y$ independent? Justify your answer.
   (d) Find $E(X - Y)$ and $\text{Var}(X - Y)$.

[20 marks]