Essentially unique: if $S, T$ minimal sufficient then $f$ 1:1 fets $q_1, q_2$ such that $T = q_1 S, S = q_2 T$

Then if $q_1 S(y) = q_1 S(y') \Rightarrow T(y) = T(y')$

$\Rightarrow q_2 T(y) = q_2 T(y') \Rightarrow S(y) = S(y')$.

Not practical to use def" to find m.s.s. - have to guess $T$, then verify the condition in def". We use instead

Theorem (1.3.0.4) (due to Lehmann & Scheffé):

Let $f(y; \theta)$ be density for $Y$. Suppose $T(Y)$ s.t. for $y$ and $x$, $f(y; \theta)$ is constant in $\theta$\n
$\frac{f(y; \theta)}{f(x; \theta)}$

iff $T(y) = T(x)$. Then $T(Y)$ is minimal sufficient.

Proof: (i) Show $T(Y)$ sufficient.

Choose and fix one element $y_t \in A_t$ for each partition set $A_t$: 

\[
\begin{array}{c}
\begin{array}{c}
A_t \subseteq A_t \\
\downarrow \\
T(y)
\end{array} \\
\begin{array}{c}
y_t's \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star \\
\star
\end{array}
\end{array}
\]

\[y = y_t \]
For any \( y \in \mathcal{Y} \), let \( y_{T(y)} \) be the fixed \( y_t \) s.t. 
\( T(y_t) = t \), in the same \( A \)-set as \( y \). Then 
\( T(y) = T(y_{T(y)}) \) and \( \frac{f(y; \theta)}{f(y_{T(y)}; \theta)} \) is indep. of \( \theta \) by assumption. Call this ratio \( h(y) \) and define 
\( g(t; \theta) = f(y_t; \theta) \) on \( \mathcal{Y}_t \), image space. 

Then 
\[
\frac{f(y; \theta)}{f(y_{T(y)}; \theta)} = \frac{f(y; \theta) f(y_{T(y)}; \theta)}{f(y_t; \theta) f(y_{T(y)}; \theta)} = h(y) g(T(y); \theta).
\]

So by the Factorization Theorem, \( T(Y) \) is sufficient.

(ii) Now show \( T(Y) \) is minimal:

Let \( T'(Y) \) be any other sufficient statistic.

By F.T., \( \exists \) functions \( g', h' \) s.t.

\[
f(y; \theta) = g'(T'(y); \theta) h'(y).
\]

Let \( y, x \) be s.t. \( T'(y) = T'(x) \)

Then 
\[
\frac{f(y; \theta)}{f(x; \theta)} = \frac{g'(T'(y); \theta) h'(y)}{g'(T'(x); \theta) h'(x)} = \frac{h'(y)}{h'(x)}
\]

which is independent of \( \theta \). By assumption

\[
\Rightarrow T(y) = T(x) \quad \text{i.e. } T(Y) \text{ is minimal sufficient.}
\]
Examples:

1. Product of Bernoulli r.v.'s

\[ Y_j \text{ i.i.d. Bernoulli}(\theta) \quad j = 1, \ldots, n \]

\[ P(Y_j = 1) = \theta, \quad P(Y_j = 0) = 1 - \theta. \]

Then

\[ \frac{f(y; \theta)}{f(x; \theta)} = \frac{\theta^k (1-\theta)^{n-k}}{\theta^{k'} (1-\theta)^{n-k'}}. \]

where \[ k = \sum_{j=1}^{n} y_j, \quad k' = \sum_{j=1}^{n} x_j \]

\[ \therefore \quad \frac{f(y; \theta)}{f(x; \theta)} = \theta^{k-k'} (1-\theta)^{(n-k)-(n-k')} \]

The ratio is constant in \( \theta \) iff \( k = k' \).

i.e., by Theorem 1.3.4, \( K = \sum_{j=1}^{n} y_j \) is minimal sufficient.

2. \( Y_j \text{ i.i.d. } \mathcal{N}(\mu, \sigma^2) \quad \mu, \sigma^2 \text{ unknown}, \quad j = 1, \ldots, n. \)

Let \( y, x \) be 2 data \( n \)-samples with sample means and variances \( (\bar{y}, s_y^2) \) and \( (\bar{x}, s_x^2) \) respectively.

Using Theorem 1.3.4,
\[
\frac{f(y; \mu, \sigma^2)}{f(x; \mu, \sigma^2)} = \left(\frac{2\pi \sigma^2}{2\pi \sigma^2}\right)^{-\frac{n}{2}} \exp\left\{ -\frac{[n(\bar{y} - \mu)^2 + (n-1)s_y^2]}{2\sigma^2} \right\} \\
\left(\frac{2\pi \sigma^2}{2\pi \sigma^2}\right)^{-\frac{n}{2}} \exp\left\{ -\frac{[n(\bar{x} - \mu)^2 + (n-1)s_x^2]}{2\sigma^2} \right\} \\
\text{check!}
\]

\[
= \exp\left\{ -\frac{n(\bar{y}^2 - \bar{x}^2) + 2n\mu(\bar{y} - \bar{x}) - (n-1)(s_y^2 - s_x^2)}{2\sigma^2} \right\} \\
\text{check!}
\]

and the ratio is independent of \((\mu, \sigma^2)\) iff \(\bar{y} = \bar{x}\)
and \(s_x^2 = s_y^2\), i.e., \((\bar{y}, s^2)\) is minimal sufficient
for \((\mu, \sigma^2)\).

3. Exponential family (scalar).

Consider
\[
\frac{f(y; \omega)}{f(x; \omega)} = \exp\left\{ s(y) \theta(\omega) - b(\omega) + c(y) \right\} \\
\exp\left\{ s(x) \theta(\omega) - b(\omega) + c(x) \right\}
\]

The ratio is constant in \(\theta\) iff \(s(y) = s(x)\),
i.e., \(s(y)\) is minimal sufficient by Theorem 10.3.4.

Notes:

(i) Once density is in exp. family form, easy to
read off the m.s.s.
(ii) Exponential family form is closed under sampling
i.e., if $Y_1, \ldots, Y_n$ are i.i.d. from a (scalar) exp.
family, then $T(Y)$ is also from a (vector) exp.
family. (See §1.2.7.)

(iii) If $Y_1, \ldots, Y_n$ are i.i.d. from an exp. family, then
so is any subset of the sufficient statistic
$T = (T_1, \ldots, T_k)$ conditional on the rest. (without
proof)
Casella & Berger, Ch 6.

§ 1.4 Completeness: no prior motivation, but ensures
uniqueness of estimators. [ACD, p. 311]

**Definition 1.4.1:** A sufficient statistic is complete if
for all (measurable) functions $g$, s.t.
$$E_\theta \{ g(T) \} = 0 \quad \forall \theta \in \Theta,$$
implies
$$P_\theta \{ g(T) = 0 \} = 1 \quad \forall \theta.$$ 

i.e., all nontrivial functions of $T$ have expectations
depending on $\theta$.

**Example:** $Y_i \sim U(0, \theta)$, $i = 1, \ldots, n$. 

Joint density $f(y; \theta) = \frac{1}{\theta^n}$, and a sufficient statistic is $T = \max(Y_i)$ (check!).

Then $P(T \leq t) = P(\text{all } Y_j \leq t) = \prod_{j=1}^{n} P(Y_j \leq t) = \left(\frac{t}{\theta}\right)^n$ (check!)

and $T$ has density $f_T(t; \theta) = \frac{nt^{n-1}}{\theta^n}$, $0 \leq t \leq \theta$.

Suppose $E\{ g(t) \} = 0 \forall \theta$.

Then $\int_0^\theta g(t) \frac{nt^{n-1}}{\theta^n} \, dt = 0 \quad \forall \theta$

$\Rightarrow \int_0^\theta g(t) t^{n-1} \, dt = 0 \quad \forall \theta$ \quad $\frac{n}{\theta^n} \neq 0$

i.e., $\int_{\theta_1}^{\theta_2} g(t) t^{n-1} \, dt = 0 \quad \forall \theta_1, \theta_2$

$= \int_0^{\theta_2} \, dt - \int_0^{\theta_1} \, dt$

i.e., $g(t) t^{n-1} = 0$

i.e., $P_\theta \{ g(T) = 0 \} = 1 \quad \forall \theta$ and $T$ is complete.
Notes:

(i) If $T$ has more components than $\Theta$, then $T$ is usually not complete.

(ii) Completeness is a property of the family of prob. dsns of the suff. stat. $T$.

(iii) $T$ from exponential families is complete.

Outline proof of (iii): Consider scalar parameter $\Theta$.

Let $f(t; \theta) = c(\theta) h(t) \exp(t\theta)$.

Suppose $E_\theta [g(T)] = 0$ for all $\theta \in \mathbb{N}$

i.e. $\int [g(t) h(t)] e^{t\theta} \, dt = 0 \quad \forall \theta \in \mathbb{N}$ (c(\theta) \neq 0)

This is the Laplace transform of $g(t) h(t) = 0$ on some open interval, so by uniqueness of LT, the inverse transform must be 0 almost everywhere

i.e. $g(t) h(t) = 0$ almost everywhere (except on set of measure 0)

i.e. $P_\Theta \{ g(T) h(T) = 0 \} = 1 \quad \forall \Theta \in \mathbb{N}$.

But $P_\Theta \{ h(T) = 0 \} = 0 \quad \forall \Theta \in \mathbb{N} \Rightarrow P_\Theta \{ g(T) = 0 \} = 1$

and $T$ is complete.
Basu's Theorem 1.4.2:

Suppose (i) \( T \) is complete and sufficient for \( \theta \), and (ii) \( V \) is any statistic with a distribution not depending on \( \theta \). Then \( T \) and \( V \) are independent random variables.

Assume \( \sigma^2 \) known, \( \mu \) unknown.

Example: \( X_i, \ i = 1, \ldots, n \) i.i.d. \( N(\mu, \sigma^2) \).

For each \( \sigma^2 \), \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \) is sufficient for \( \mu \), and is also complete. \( S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \) has a dsn not depending on \( \mu \) (what is it?) \( \sigma^2 \chi^2_{n-1} \).

i.e. \( \bar{X} \) and \( S^2 \) are independent for each \( \sigma^2 \), i.e true for all \( \sigma^2 \).

Proof of Theorem: (cont. case)

Fix any \( \sigma^2 \), then \( P_{\theta} (V \leq v) = q_v \), say, independent of \( \theta \) by (ii). Also, \( T \) is sufficient, so \( P_{\theta} (v \leq v \mid T) \) is a fct of \( T \), indep. of \( \theta \).

So consider \( E_{\theta} \left\{ P_{\theta} (V \leq v \mid T) - q_v \right\} \)

\[ = E_{\theta} \left\{ E_{\theta} (I_{\{V \leq v\}} \mid T) - q_v \right\} \]
= E_θ \{ I_{\{V \leq T\}} \} - q_v = P(V \leq T) - q_v = 0 \quad \forall \theta

But T is complete, so P_θ(V \leq T) = q_v with probability 1, i.e. P_θ(V \leq T) = P_θ(V \leq U).

i.e. V and T are independent r.v.s.

Remarks:

1. See ACD p. 649 for proof in discrete case.

2. "Converse": (i) + conclusion \( \Rightarrow \) (ii).

Snag: it can be harder to show a statistic is complete than to show the independence of 2 statistics!