Chapter 1: Modern Statistical Inference

§ 1.1 Moments, cumulants and the cumulant generating function

[Reference: § 2.4 'Statistical Models' by A.C. Davison, hereafter we refer to as ACD.]

1.1.1 Properties of distributions:

**Expectation:**

\[
E \left[ g(x) \right] = \begin{cases} 
\int_{-\infty}^{\infty} g(x) f_x(x) \, dx & \text{(continuous)} \\
\sum_{\text{all } x} g(x) P(X=x) & \text{(discrete)}
\end{cases}
\]

provided the integral/sum converges absolutely.

**Examples:**

(a) \( E \left[ X^n \right] = \mu^n \) - \( n \)th moment about the origin
\[ E[ (X-\mu)^r ] = \mu_r \quad \text{r\textsuperscript{th} moment about mean (central moment)} \]

\[ \mu'_1 = \mu = E[X] = \text{mean (if it exists)} \]

\[ \mu'_2 = E[ (X-\mu)^2 ] = \sigma^2 = \text{variance} \]

(b) \[ M_X(t) = E[ e^{tx} ] \quad \text{provided the fat. exist for } t \in \text{ an open neighbourhood of the origin} \]

**Linear Properties of \( E[ ] \):**

\[ E[ a_1 g_1(x) + a_2 g_2(x) + a_0 ] \]

\[ = a_1 E[ g_1(x) ] + a_2 E[ g_2(x) ] + E[ a_0 ] \]

\[ = a_0 + a_1 E[ g_1(x) ] + a_2 E[ g_2(x) ] \quad \text{etc} \]

Then

\[ E[ e^{tx} ] = E[ 1 + tx + \frac{1}{2!} t^2 x^2 + \frac{1}{3!} t^3 x^3 + \ldots ] \]

\[ = 1 + t\mu + \frac{1}{2!} t^2 \mu'_2 + \frac{1}{3!} t^3 \mu'_3 + \ldots \]

\[ \Rightarrow \mu'_k = \frac{d^k M_X(t)}{dt^k} \bigg|_{t=0} \quad \text{assuming validity of diff. under sign w.r.t. } t. \]

i.e. its derivatives at origin, \( t=0 \), are the moments of the c.d.s.n.
Characteristic Function: of a univariate r.v. $X$ is a complex valued fct of the real variable $u$ defined by

$$
\psi_X(u) = E\left[e^{iuX}\right].
$$

Unlike the MGF, the char. fct. always exists, and determines the dsn. fct $F_X(x)$ uniquely.

Theorem 11.2 (without proof):

If $M_X(t)$ exists, then $\psi_X(u) = M_X(iu)$.

Cumulants: Set of constants associated with the dsn; equivalent to moments, but often simpler to deal with.

Definition 11.3: The cumulant generating function for the dsn of a r.v. $Y$ is

$$
K_Y(t) = \log M_Y(t).
$$

If the CGF possesses a power series expansion about $t=0$, then
\[ K_y(t) = \kappa_1 t + \frac{1}{2!} \kappa_2 t^2 + \cdots + \frac{1}{r!} \kappa_r t^r + \cdots \]

\[ = \sum_{r=1}^{\infty} \frac{\kappa_r t^r}{r!} \]

where \( \kappa_r = \frac{d^r}{dt^r} \left. K(t) \right|_{t=0} = K^{(r)}(0) \);

\( \kappa_r \) = \( r \)th coeff. of \( t^r \) is called the \( r \)th cumulant.

**Lemma 1.1.4:**

\( \kappa_1 = \mu \), \( \kappa_2 = \sigma^2 \), \( \kappa_3 = \text{skewness} \),

\( \kappa_4 = \text{kurtosis} \), ...

**Proof:** \[ \text{Differentiate } K_y(t) \]

\[ \frac{d K_y(t)}{dt} = \frac{1}{\Pi_y(t)} \frac{d \Pi_y(t)}{dt} \]

\( \text{Let } t=0 \Rightarrow \left. \frac{d K_y(t)}{dt} \right|_{t=0} = \frac{1}{\Pi_y(0)} \left. \frac{d \Pi_y(t)}{dt} \right|_{t=0} \]

\( = \mu = \mu'_1 = \kappa_1 \).

\[ \text{Differentiate again,} \]

\[ \frac{d^2 K_y(t)}{dt^2} = \frac{1}{\Pi_y(t)} \frac{d^2 \Pi_y(t)}{dt^2} - \frac{1}{(\Pi_y(t))^2} \left( \frac{d \Pi_y(t)}{dt} \right)^2 \]

\( \text{Let } t=0 \Rightarrow \left. \frac{d^2 K_y(t)}{dt^2} \right|_{t=0} = \mu'_2 - (\mu'_1)^2 = \kappa_2. \)
We know that
\[ E \left[ (X - \mu)^2 \right] = E[X^2] - [E(X)]^2 \]
\[ = \mu^2 - (\mu_1)^2 \quad \text{check!} \]
\[ = \sigma^2. \]

In general, for \( r > 0 \),
\[ \kappa_r = \mu' + \{ \text{polynomial terms involving } \mu_1, \ldots, \mu_{r-1} \} \]
and these expressions may be taken as alternative definitions for the cumulants that avoid explicit use of the CGF.

[See Assignment 1.]

Example: Let \( Y \sim N(\mu, \sigma^2) \). We know its MGF is
\[ M_Y(t) = \exp \left( t\mu + \frac{1}{2}t^2\sigma^2 \right) \]
\[ \Rightarrow \kappa_Y(t) = t\mu + \frac{1}{2}t^2\sigma^2. \]

Then \( \kappa_1 = \mu \), \( \kappa_2 = \sigma^2 \), and all its higher-order cumulants are zero.

Suppose \( Z \sim N(0,1) \). Then \( \kappa_Z(t) = \frac{1}{2}t^2 \)