Twistor Theory.

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1 Introduction

Twistor theory began with the work of Roger Penrose who introduced the powerful techniques of complex algebraic geometry into general relativity. Loosely speaking it is the use of complex analytic methods to solve problems in real differential geometry. In most cases the emphasis is on the geometry of the problem rather than the analysis. My lectures are not designed to be a survey of all of twistor theory or even the most important parts but will concentrate instead on those areas with which I am most familiar.

I will first the so-called mini-twistor space of $\mathbb{R}^3$ and how it can be used to find harmonic functions, or functions in the kernel of the Laplacian, by an integral transform. This intergral transform is, in fact, a classical result given in Whittaker and Watson's famous book [21]. Also in there is an integral formula for solutions of the wave equation on $\mathbb{R}^4$. I will show how this also fits in the context of twistor theory by giving a general twistor integral transform — called a Penrose transform — for the solution of any constant coefficient homogeneous linear differential equation. To present the general result requires a certain amount of complex geometry in particular the theory of complex line bundles. This will be explained as I go along. A good reference is [9].

Twistor theory can also be used to solve non-linear diferential equations which are related to the self-duality equations that describe instantons in $\mathbb{R}^4$. I will present a brief account of the theory of Bolgomolny equations. These are essentially time-invariant instantons and the twistor correspondence uses the mini-twistor space I described at the beginning.

There are many other topics in twistor theory. For example there are twistor methods for defining hyper-kaehler manifolds [2] and applications of twistor theory to the representation theory of Lie groups [4]. Many short notes on twistor theory have traditionally appeared in the informal 'Twistor Newsletter'. Various compendiums of these are available and I have listed them in the bibliography [14, 15, 16]

2 Minitwistors

It has been known for a long time that problems in real differential geometry can often by simplified by using complex co-ordinates. For example in the plane $\mathbb{R}^2$ we can write $z = x + iy$ and thereby identify $\mathbb{R}^2$ with $\mathbb{C}$. We then discover that a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic if and only if we can write it as

$$f = \psi + \bar{\psi}$$

where $\psi\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic (complex-analytic) function. So we have related harmonic, real-valued functions – a problem in real differential geometry on the plane – to holomomorphic functions of one-variable – a problem in complex analysis.

If we try the same technique in $\mathbb{R}^3$ we discover the unfortunate fact that it has odd dimension and hence cannot be identified with some $\mathbb{C}^n$! We can however form another space closely associated to the geometry of $\mathbb{R}^3$ that is intrinsically complex. This is the space Z of all oriented lines in $\mathbb{R}^3$ known to twistor theorists as *mini-twistor space*. The word 'mini' here refers to the fact that Penrose's original twistor theory was defined in four dimensions. Notice that any oriented line ℓ in $\mathbb{R}^3$ is determined uniquely by giving the unit vector u parallel to the line in the direction of the orientation and the shortest vector v joining the origin to ℓ. The vector v is well-known to be determined by the fact that is orthogonal to u. We then have

$$\ell = \{v + tu \mid t \in \mathbb{R}\}.$$

This shows that

$$Z = \{(u, v) \mid u, v \in \mathbb{R}^3, \|u\|^2 = 1, \langle u, v\rangle = 0\} \subset S^2 \times \mathbb{R}^3.$$

The mini-twistor space Z is readily seen to be TS^2 the tangent bundle of the two sphere, that is the space formed by taking the union of all the tangent planes to the two-sphere or

$$TS^2 = \cup_{u \in S^2} T_u S^2$$

where

$$T_uS^2 = \{v \in \mathbb{R}^3 \colon \langle u, v\rangle = 0\}$$

is the tangent plane to the two-sphere at u. Notice that there is a projection map $Z \to S^2$ sending a line to its direction or a pair (u, v) to v.

Whereas $\mathbb{R}^3$ is three dimensional the mini-twistor space Z is four dimensional and now we have a chance of identifying it with something complex. Of course it is not a Euclidean space but a manifold so we have to consider complex manifolds.

Recall that a (real) manifold M of dimension n is essentially a set which can be covered by co-ordinate charts in such a way that if co-ordinates $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$ have overlapping domains then on that overlap we have

$$x^i(p) = F^i(y(p))$$

for some $F(y^1, \dots, y^n)$ which is a smooth (that is infinitely differentiable) function of n real variables. We call F the change of co-ordinates map. To remove the 'essentially' and give a precise definition I would only need to be a little more careful about the domain of definition of x, y and F. I refer the reader to [10] or any other book on differential geometry.

A complex manifold is the obvious generalisation of this defintion. It has complex valued co-ordinates and the corresponding co-ordinate change maps like F are required to be holomorphic. Being holomorphic is a stronger condition than infinite differentiability and the theory of complex manifolds has a rich structure because of this. A good reference is [9]. Instead of making the definition anymore precise I will illustrate it with an example. The example is the simplest, non-trivial complex manifold – the two-sphere.

Example 2.1 (The two-sphere). Define two open subsets U_0 and U_1 of S^2 by removing the north and south poles of the two-sphere. That is

$$U_0 = S^2 - \{(0,0,1)\} \quad \text{and} \quad U_0 = S^2 - \{(0,0,-1)\}.$$

These will be the domain of our co-ordinates. The union of U_0 and U_1 clearly covers all of S^2. We define complex co-oordinates on U_0 by

$$\xi_0(x, y, z) = \frac{x + iy}{1 - z}$$

and on U_1 by

$$\xi_1(x, y, z) = \frac{x - iy}{1 + z}.$$

Geometrically the first set of co-ordinates is defined by stereographic projection from $(0, 0, 1)$. That is if we draw a line through $(0, 0, 1)$ and (x, y, z)

then it meets the XY plane at $(x/(1-z), y/(1-z))$. The second set is defined by stereographic projection from $(0,0,-1)$ with a change of sign in the y co-ordinate. The change of sign is so that the co-ordinate change map is holomorphic. If we do not change the sign we find that the co-ordinate change map is anti-holomorphic, that is of the form $w \mapsto \bar{F}(w)$ where F is holomorphic. We calculate the co-ordinate change map to be

$$\xi(x,y,z) = \frac{1}{\xi_1(x,y,z)} = F(\xi_1(x,y,z))$$

and see that it is the holomorphic function $F(w) = 1/w$.

In the case of $\mathbb{R}^2$ harmonic functions h are generated by holomorphic functions ψ by the simple device of writing

$$h(x,y) = \psi(x+iy) + \bar{\psi}(x+iy).$$

This exploits the fact that the map $(x,y) \mapsto x+iy$ identifies the space $\mathbb{R}^2$ with the space $\mathbb{C}$. To construct harmonic functions on $\mathbb{R}^3$ we need to do something more complicated because the spaces $\mathbb{R}^3$ and Z cannot be identified. There are lots of reasons for this, for example they do not have the same topology, $\mathbb{R}^3$ can be contracted to a point and Z can only be contracted to a two-sphere, but it is simplest to note that they have different real dimensions, three and four respectively. To understand the relationship between $\mathbb{R}^3$ and Z we have to explore their geometry a little further.

By definition the points of Z are oriented lines in $\mathbb{R}^3$. Moreover any point in $\mathbb{R}^3$ defines a two-spheres worth of lines, all the oriented lines going through that point. If the point in question is p then the set of all lines through p is the set of all (u,v) satisfying

$$v = p - \langle p, u\rangle u.$$

We shall call this subset a real section of Z and denote it by X_p. I need to explore further the geometry of these real sections.

Firstly let me explain why they are called sections. Notice that the map

$$\begin{array}{rccc} \rho_p: & S^2 & \to & Z \\ & u & \mapsto & (u, v = p - \langle p, u\rangle u) \end{array}$$

defines a section of the projection $Z \to S^2$. (Recall that a section of a projection $\pi\colon X \to Y$ is a map $\rho\colon Y \to X$ with $\pi(\rho(y)) = y$ for all $y \in Y$.) The image of this section is X_p. I am, of course, abusing language here as I am calling the X_p a real section rather than the image of a real section – I will continue to do this.

Next we need a map

$$\tau\colon Z \to Z,$$

called the real structure. It is defined to be the involution that sends a line with orientation to the same line with opposite orientation, that is $\tau(u, v) = (-u, v)$. The real structure fixes the set X_p because

$$\tau(u, p - \langle p, u\rangle u) = (-u, p - \langle p, u\rangle u) = (-u, p - \langle p, -u\rangle - u).$$

This is the reason the section X_p is called real.

Finally the map ρ_p is holomorphic, in fact given by a quadratic polynomial. To see this we need to consider the complex structure of Z which we have avoided up to know. The complex co-ordinates we have defined above on the two-sphere can be used to define complex co-ordinates (η, ζ) on Z. Let $\widetilde{U}_0$ be the set of all $(u, v) \in Z$ where $u \in U_0$. Then we define

$$\zeta(u, v) = \frac{u_1 + iu_2}{1 - u_3}. \tag{1}$$

Notice that this is just the co-ordinates on S^2 applied to u, the point v has played no role yet. To obtain complex co-ordinates we differentiate the map ζ. We obtain

$$\eta(u, v) = \frac{v_1 + iv_2}{1 - u_3} + \frac{(u_1 + iu_2)v_3}{(1 - u_3)^2}. \tag{2}$$

If $p = (x, y, z)$ is a point in $\mathbb{R}^3$ then we have defined X_p to be the set of all (u, v) that correspond to lines through p. This is the set

$$X_p = \{(u, p - \langle u, p\rangle u) \mid u \in S^2\}.$$

If we substitute $v = p - \langle u, p\rangle u$ into the equation for η and simplify we see that the equation defining X_p as a subset of Z is

$$\eta = \frac{1}{2}((x + iy) + 2z\zeta - (x - iy)\zeta^2) \tag{3}$$

where (x, y, z) are the co-ordinates of p.

Hence, in local co-ordinates, the map ρ_p is

$$\rho_p(\zeta) = (\frac{1}{2}((x + iy) + 2z\zeta - (x - iy)\zeta^2), \eta). \tag{4}$$

We call a section that can be written locally as a holomorphic function as in equation (4) a holomorphic section. It is possible to show (see 5) that all holomorphic sections $S^2 \to Z$ take the form

$$\zeta \mapsto (\zeta, a + b\zeta + c\zeta^2)$$

in co-ordinates, where a, b and c are complex numbers. With our choice of co-ordinates and the definition of the real structure it is an easy exercise to see that if (η, ζ) are the co-ordinates of p then $(-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$ are the co-ordinates of $\tau(p)$. So τ is an anti-holomorphic map. A section is therefore real if the equation

$$\eta = a + b\zeta + c\zeta^2$$

defines the same subset of Z as the equation

$$-\frac{\bar{\eta}}{\bar{\zeta}^2} = a + b\frac{-1}{\bar{\zeta}} + c\frac{1}{\bar{\zeta}^2}.$$

Simplifing we see that this is true if and only if $a = -\bar{c}$ and b is real. Hence the real sections ρ_p defined by points in $\mathbb{R}^3$ are precisely all the real sections of Z. So we have a bijection between points of $\mathbb{R}^3$ and real holomorphic sections of Z.

The correspondence we have now established between $\mathbb{R}^3$ and Z is completely symmetric; points in Z define special subsets (oriented lines) in $\mathbb{R}^3$ and points in $\mathbb{R}^3$ define special subsets (holomorphic real sections) in Z.

Consider a differential form $\omega = g(\eta, \zeta)d\eta$ on Z. We can use this to define a function on $\mathbb{R}^3$ by

$$\phi(x, y, z) = \int g(((x + iy) + 2z\zeta - (x - iy)\zeta^2)/2, \zeta)d\zeta, \tag{5}$$

It is easy to differentiate through the integral sign in 5 and see that

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0.$$

So the function ϕ is harmonic. This result is a classical integral formula contained in Whittaker and Watson [21]. To obtain their formulation we replace the integral in η by a contour integral around the circle and let $\eta = -\exp(iw)$ to obtain

$$\phi(x, y, z) = \int_{-\pi}^{\pi} h(z + ix\cos(w) + iy\sin(w), w)dw$$

where $h(v) = g(\exp(iw)v$.

Notice that instead of performing and integral in equation (5) over S^2 we can think of this as being first the restriction of ω to X_p and second its integral over X_p so that we have

$$\phi(p) = \int_{X_p} \omega.$$

3 A classical result

Whittaker and Watson also give an integral formula for the solutions of the wave equation on $\mathbb{R}^4$. It is

$$\phi(x,y,z,t) = \int_{-\pi}^{\pi} g(x\sin(w)\cos(w') + y\sin(w)\sin(w') + z\cos(w) + t, w, w')dwdw'.$$

Again it is easy to check that this satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\phi}{\partial t^2} = 0.$$

I want to explain how these two classical examples form part of a general Penrose transform or twistor correspondence. This Penrose transform will give rise to integral formulae for the solutions of any homogeneous real-valued polynomial differential equation. That is any differential equation for functions ϕ on $\mathbb{R}^{n+1}$ of the form

$$D_f\phi = f(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n})\phi(x^0, \dots, x^n) = 0 \tag{6}$$

where f is a homogeneous real-valued polynomial of degree $k > 1$ and $n > 1$. I will show that there is a twistor space Z, a vector space $H^{n-1}(Z, \mathcal{O}(-n-1+k))$ of Dolbeault cohomology classes (think of them for now as differential forms) and a twistor correspondence

$$T\colon H^{n-1}(Z, \mathcal{O}(-n-1+k)) \to C^\omega(\mathbb{R}^{n+1}, \mathcal{O})$$

where C^ω denotes the real-analytic functions. This is an injective map, defined by integrating the differential form, onto the space of all the real-analytic functions in the kernel of D_f if $k \leq n$.

The flavour of this correspondence is as follows. We start with a polynomial f and $\mathbb{R}^{n+1}$. There is a twistor space Z which depends on f, and has a family of submanifolds X_p indexed by elements p in $\mathbb{R}^{n+1}$. In the minitwistor space case we integrated differential forms over the X_p. In the more general case we integrate so-called differential forms ρ 'of type $(0, n-1)$ with values in the line bundle $\mathcal{O}(-n-1+k)$', or their cohomology classes. The result then is a function

$$T(\rho)(p) = \int_{X_p} \omega$$

on $\mathbb{R}^{n+1}$.

To explain this correspondence in more detail I have to explain the differential geometry of Z, why these differential forms ρ are the correct thing to integrate over X_p and what the cohomology space $H^{n-1}(Z, \mathcal{O}(-n-1+k))$ is. The spaces X_p are all isomorphic to a subvariety X of complex projective space defined by f so we start then with the geometry of complex projective space.

4 Complex projective space

Recall that n dimensional complex projective space is the n dimensional complex manifold $\mathbb{C}P_n$ of all lines through the origin in $\mathbb{C}^{n+1}$. If z is a non-zero vector in $\mathbb{C}^{n+1}$ then we denote by $[z]$ the set of all non-zero multiples of z that is

$$[z] = \{\lambda z \mid \lambda \in \mathbb{C} - \{0\}\}.$$

Notice that $[z]$ is the line through z with the origin or 0 removed. Clearly we can identify the line through z with $[z]$ so we identify the elements of $\mathbb{C}P_n$ with the set of all $[z]$. Indeed I will often refer to $[z]$ as a line. It follows from this definition that

$$[z^0, \ldots, z^n] = [\lambda z^0, \ldots, \lambda z^n]$$

for any non-zero complex number λ. If $[z]$ is a line then the numbers $(z^0, \ldots, z^n)$ are called the homogeneous co-ordinates of the line. Note that the homogeneous co-ordinates of a line are not unique although we can uniquely specify things like ratios of the z^i when these are finite.

We will need to know below that $\mathbb{C}P_n$ is compact as a topological space. To see this notice that the $2n-1$ sphere in $\mathbb{C}^n = \mathbb{R}^{2n}$ is the set of all z such that $\sum_{i=1}^n |z^i|^2 = 1$. If we map each such unit vector to the line containing it we define a continous, onto map

$$S^{2n-1} \to \mathbb{C}P_n.$$

The sphere is, of course, compact and hence $\mathbb{C}P_n$ is compact.

We make $\mathbb{C}P_n$ a complex manifold by considering the open subsets U_i where the ith homogeneous co-ordinate is non-zero. On these we can define co-ordinates

$$[z^0, \ldots, z^n] \mapsto (\frac{z^0}{z^i}, \ldots, \frac{z^{i-1}}{z^i}, \frac{z^{i+1}}{z^i} \ldots, \frac{z^n}{z^i}) \tag{7}$$

taking values in $\mathbb{C}^n$.

Example 4.1 ($\mathbb{C}P_1$). We have already met one-dimensional complex projective space. It is just the two-sphere. An explicit diffeomorphism between S^2

and $\mathbb{C}P_1$ is given by the map

$$\begin{array}{ccc} S^2 & \to & \mathbb{C}P_1 \\ (x, y, z) & \mapsto & [x + iy, 1 - z]. \end{array}$$

We leave it as an exercise for the reader to check that the co-ordinates defined in Section 2 are essentially the same as the co-ordinates defned in equation (7) when composed with this diffeomorphism.

If f is a homogenous polynomial of degree k then we can apply it to the homogeneous co-ordinates of a line and we have, of course

$$f(\lambda z^0, \ldots, \lambda z^n) = \lambda^k f(z^0, \ldots, z^n).$$

It follows that although f applied to a line $[z]$ is not well defined, we can uniquely define the subset X of $\mathbb{C}P_n$ where f vanishes. We just define this to be the set of $[z]$ where $f(z) = 0$. In general X is a projective algebraic variety, for convenience we will assume that it a smooth submanifold of $\mathbb{C}P_n$.

Example 4.2 (Mini-twistors). For the case of the Laplacian in $\mathbb{R}^3$ we are interested in the polynomial

$$f(x, y, z) = x^2 + y^2 + z^2.$$

The variety X in $\mathbb{C}P_2$ is the quadric defined to be the set of all $[z^0, z^1, z^2]$ satisfying

$$(z^0)^2 + (z^1)^2 + (z^2)^2 = 0.$$

This is isomorphic to $\mathbb{C}P_1$ or S^2 via the map

$$w \mapsto [i(w^2 + 1), (w^2 - 1), 2w].$$

On the open set U_i the set where f vanishes is the zero set of the polynomial

$$f(\frac{z^0}{z^i}), \ldots, 1, \ldots, \frac{z^n}{z^i})$$

but this polynomial does not extend to a nice function on all of $\mathbb{C}P_n$. There is a good reason for this which I wish to discuss next as it is also the reason that we have to introduce holomorphic line bundles in the next section.

If we have a complex manifold M then it makes sense to talk about holomorphic functions $\chi\colon M \to \mathbb{C}$, just as for the case of smooth functions on a real manifold. We just define a function $\chi\colon M \to \mathbb{C}$ to be holomorphic if for every set of co-ordinates $z^1, \ldots, z^n$ it can be written as

$$\chi(m) = g(z^1(m), \ldots, z^n(m))$$

for some holomorphic function $g(z^1, \dots, z^n)$. However if $\chi\colon M \to \mathbb{C}$ is a holomorphic function on a compact complex manifold then it must be constant. This is because of the maximum modulus theorem. Indeed the function $|\chi|^2$ is a continous function on the compact space M so has a maximum at some point m. But we can now choose an open set about m and introduce complex co-ordinates to make that set look like an open subset of $\mathbb{C}^n$. But a holomorphic function whose modulus has a maximum in the interior of an open set is constant. It follows that to obtain a useful function theory on compact complex manifolds we have to do something more than just look at globally defined functions. There are two generalisations that are possible, the first is to sections of holomorphic line bundles and the second is to sheaves. We shall need to only consider holomorphic line bundles for the purposes of these lectures and we will do that in the next section 5.

The integral transform we wish to construct takes the form:

$$T(\omega)(p) = \int_{X_p} \omega$$

and I have been claiming that ω is a type of differential form. I now want to consider precisely what kind of differential form it is.

Recall that if M is an oriented real manifold of dimension m then we can integrate over it differential forms of degree m. In the case that M is a complex manfiold of dimension n with co-ordinates $z^1, \dots, z^n$ we can integrate complex differential forms of the type:

$$g(z, \bar{z}) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n. \tag{8}$$

The reason for this is that if we let $z^j = x^j + iy^j$ we obtain co-ordinates (x^j, y^j) for M thought of as an $m = 2n$ dimensional real manifold. If we then use $dz^j = dx^j + idy^j$ and $d\bar{z}^j = dx^j - idy^j$ and expand out equation (8) becomes a differential form which is a multiple of

$$dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n$$

and this is something we can integrate.

More general differential forms on a complex manifold can also be decomposed into products of dz's and $d\bar{z}$'s. We say a general form is of type (p, q) if it has p dz's and q $d\bar{z}$'s. We will be particularly interested in forms of type $(0, n)$. Indeed we will adopt the rather perverse notion that the forms like

$$g(z, \bar{z}) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.$$

should be written as

$$[g(z, \bar{z}) dz^1 \wedge \cdots \wedge dz^n]\ d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.$$

and thought of, not as forms of type (n, n), but as forms of type $(0, n)$ with values in the *holomorphic line bundle* of all forms of type $(n, 0)$. We denote by K_m the space of all forms at $m \in M$ of type $(n, 0)$. In other words K_m is the highest exterior power of the cotangent space T_pM^*. It follows that K_m is 1 dimensional. The union of all these one dimensional spaces as m varies is called the *canonical line bundle* of M. The moral of the story then is that on a complex manifold of dimension M we can integrate $(0, n)$ forms with values in the canonical line bundle K.

More generally we will be interested in other holomorphic line bundles over a complex manifold M so in the next section we will discuss the theory of these.

5 Holomorphic line bundles

We have seen that on a compact complex manifold the only holomorphic functions are the constants. We want to introduce the concept of a holomorphic section of a holomorphic line bundle to gives us a larger collection of functions like objects to work with. To do this we associate to every point of a complex manifold M a one-dimensional complex vector space L_m. Denote by L the union of all these spaces. Note that in principle the L_m's are all different but not very different. A one-dimensional complex vector space is *nearly* the same as $\mathbb{C}$. Indeed if we pick any non-zero vector v in L_m then it defines a basis of L_m and hence a linear isomorphism

$$\begin{array}{ccc} \mathbb{C} & \to & L_m \\ z & \mapsto & xv \end{array}$$

Instead of holomorphic functions we want to consider holomorphic sections of L. A section is a functions

$$\psi\colon M \to L = \cup_{m \in M} L_m$$

with the property that $\psi(m) \in L_m$ for every m. Notice that because L_m is a vector space we can add sections by defining

$$(\psi + \chi)(m) = \psi(m) + \chi(m)$$

and multiply them by scalars by defining

$$(z\psi)(m) = z\psi(m).$$

So the collection of all sections forms a vector space and they behave in many ways like functions.

We want to consider holomorphic sections and for that we need a notion of a family of vector spaces L_m varying holomorphically with m. To make sense of this we need to consider the precise definition of a holomorphic line bundle.

Definition 5.1 (Holomorphic line bundle.)**.** A holomorphic line bundle over a manifold M is a holomorphic manifold L with a holomorphic map $\pi\colon L \to M$ such that:

1. each fibre $L_m = \pi^{-1}(m)$ is one-dimensional complex vector space, and
2. we can cover M with open sets U_α such that there is a bi-holomorphic map $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}$ with the property that for all $m \in U_\alpha$ we have $\psi_\alpha(L_m) \subset \{m\} \times \mathbb{C}$ and moreover $\psi_{\alpha|L_m}$ is a linear isomorphism.

In the second part of this definition that requires that

$$\psi_{\alpha|L_m}\colon L_m \to \{m\} \times \mathbb{C}$$

is a linear isomorphism we make $\{m\} \times \mathbb{C}$ into a vector space by defining

$$\lambda(b, z) + \mu(b, w) = (b, \lambda z + \mu w).$$

A bi-holomorphic map is just an invertible holomorphic map whose inverse is also invertible. We call the set $L_m = \pi^{-1}(m)$ the fibre of L over m.

We can now define

Definition 5.2 (Holomorphic section.)**.** If $L \to M$ is a holomorphic line bundle then a holomorphic section of L is a holomorphic map $\psi\colon M \to L$ such that $\psi(m) \in L_m$ for all $m \in M$.

Example 5.1 (The trivial bundle). The simplest example of a holomorphic line bundle is the *trivial* bundle $L = M \times \mathbb{C}$. In this case it is easy to see that a section is just a holomorphic function. Indeed a section must have the form $\psi(m) = (m, \chi(m))$ for some holomorphic map $\chi\colon M \to \mathbb{C}$.

It is often useful to have a 'local' description of sections. To do this we note from the definition that we can cover M with open sets $\{U_\alpha\}$ such that there are local non-vanishing holomorphic sections $\psi_\alpha\colon U_\alpha \to L$. At any point $m \in U_\alpha \cap U_\beta$ we now have two non-zero elements $\psi_a(m)$ and $\psi_\beta(m)$ in the one-dimensional vector space L_m. These must differ by a scalar $g_{\alpha\beta}(m)$ and hence there are holomorphic maps

$$g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C} - \{0\}$$

(called transition functions) such that

$$\psi_\alpha = \psi_\beta g_{\alpha\beta}. \tag{9}$$

on $U_\alpha \cap U_\beta$.

Notice that to be completely precise we should write equation (9) as

$$\psi_{\alpha|U_\alpha \cap U_\beta} = \psi_{\beta|U_\alpha \cap U_\beta} g_{\alpha\beta}$$

but we shall usually drop the notation that indicates the restriction of functions to subsets such as $U_\alpha \cap U_\beta$.

Any section ξ of L can then be written at points of each U_α as

$$\xi = \xi_\alpha \psi_\alpha.$$

The section ξ is holomorphic if and only if each of the

$$\xi_\alpha \colon U_\alpha \to \mathbb{C}$$

is holomorphic. The ξ_α must satisfy

$$\xi_\alpha g_{\alpha\beta} = \xi_\beta$$

on the intersection $U_\alpha \cap U_\beta$. A converse to this result is also true. If we can find holomorphic functions $\xi_\alpha \colon U_\alpha \to \mathbb{C}$ such that

$$\xi_\alpha g_{\alpha\beta} = \xi_\beta$$

on the intersection $U_\alpha \cap U_\beta$ then defining $\xi = \xi_\alpha \psi_\alpha$ on each open set U_α defines a global holomorphic section of L.

Example 5.2 (The tautological bundle)*.* We now define a natural holomorphic line bundle H over complex projective space called the *tautological line bundle*. The fibre of H over a point $[z] \in \mathbb{C}P_n$ is just the line $[z]$. To see how all the fibres fit together we define

$$H = \{([z], w) \mid w = \lambda z, \lambda \in \mathbb{C} - \{0\}\} \subset \mathbb{C}P_n \times \mathbb{C}^{n+1}.$$

The projection map $H \to \mathbb{C}P_n$ is just the restriction of the obvious projection from $\mathbb{C}P_n \times \mathbb{C}^{n+1}$, that is $([z], w) \mapsto [z]$. We sdefine local sections $\psi_i \colon U_i \to H$ for each $i = 0, \ldots, n$ by

$$\psi([z]) = ([z], (\frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i})).$$

Note that they do have image in H because

$$(\frac{z_0}{z_i}, \ldots, 1, \ldots, \frac{z_n}{z_i}) = \frac{1}{z_i}(z_0, \ldots, z_n).$$

From the definition of the ψ_i we have

$$\psi_i([z]) = \frac{1}{z_i}(z) = \frac{z_j}{z_i}\psi_j([z]).$$

and hence the transition functions are

$$g_{ij} = \frac{z_j}{z_i}.$$

Example 5.3 (The tangent bundle to S^2). In the case of $\mathbb{C}P_1$ the tangent space at any point is a one-dimensional complex vector space and hence is a complex line bundle. Notice that elementary topology tells us that it cannot be a trivial line bundle. If it was it would have a non-vanishing (holomorphic) section and therefore TS^2 would have a non-vanishing section. But this means that the two-sphere would have a non-vanishing continous tangent vector and this is not possible [18].

Operations on vector spaces such as taking duals and forming tensor products can also be applied, fibre by fibre, to a line bundle. So we define L^* the dual of the line bundle L by $(L^*)_m = (L_m)^*$ and $L \otimes J$, the tensor product of the line bundles L and J by $(L \otimes J)_m = L_m \otimes J_m$. If v is a non-zero element of a one dimensional vector space it naturally defines an element v^{-1} of the dual space by the requirement that $v^{-1}(v) = 1$. Non-vanishing local sections ψ_α of L hence give rise to non-vanishing sections ψ_α^{-1} of L^*. Similarly local sections of L and J can be tensored together to give local sections of $L \otimes J$. Notice that if $g_{\alpha\beta}$ are transition functions for L and $h_{\alpha\beta}$ are transition functions for J then the transition functions for L^* are $g_{\alpha\beta}^{-1}$ and those for $L \otimes J$ are $g_{\alpha\beta}h_{\alpha\beta}$.

We can now define

Definition 5.3 (Isomorphism of line bundles)**.** Two line bundles L and J are said to be *isomorphic* if at each point m of M we have a linear isomorphism

$$\alpha_m \colon L_m \to J_m.$$

such that the induced section of $L^* \otimes J$ is holomorphic.

Example 5.4 (The bundles $\mathcal{O}(p)$). Returning to the line bundle H over $\mathbb{C}P_n$ we can now define a family of line bundles over $\mathbb{C}P_n$ denoted by $\mathcal{O}(p)$ for any integer p as follows. For p a negative integer we define $\mathcal{O}(p)$ to be the $-p$th tensor power of H with itself and for p a positive integer we define $\mathcal{O}(p)$ to be the pth tensor power of H^* with itself. These line bundles satisfy; $\mathcal{O}(-1) = H$, $\mathcal{O}(p) \otimes \mathcal{O}(q) = \mathcal{O}(p+q)$, $\mathcal{O}(p)^* = \mathcal{O}(-p)$ and $\mathcal{O}(0)$ is the trivial line bundle.

The reason for the choice of sign here is that the the line bundle H has no holomorphic sections except for the zero section. Let us see why this is true in the case of $\mathbb{C}P_1$. There we have two open sets U_0 and U_1 and a holomomorphic section of H consists of a pair of functions $\xi_0 \colon U_0 \to \mathbb{C}$ and $\xi_1 \colon U_1 \to \mathbb{C}$ such that

$$\xi_0([z]) = \frac{z_1}{z_0}\xi_1([z]).$$

To understand this let us translate it into co-ordinates. Let $\zeta([z]) = z_0/z_1$ then we require two holomorphic functions ξ_0 and ξ_1, defined on all of $\mathbb{C}$, such that

$$\xi_0(\zeta) = \frac{1}{\zeta}\xi_1(\frac{1}{\zeta}).$$

If we Taylor expand both sides of this equation in ζ and $1/\zeta$ we deduce that $\zeta_0 = \zeta_1 = 0$ is the only solution. Similarly if we try and find a holomorphic section of $\mathcal{O}(p)$ we seek holomorphic functsions ξ_0 and ξ_1 defined on all of $\mathbb{C}$ such that

$$\xi_0(\zeta) = \zeta^p\xi_1(\frac{1}{\zeta}).$$

Again Taylor expanding shows that if p is negative there are no non-zero sections and if p is positive there is a vector space of sections of dimension $p+1$.

Notice that this gives us a simple interpretation of the number p in the case of a line bundle $\mathcal{O}(p)$ over $\mathbb{C}P_1$ with $p \geq 0$. That is, it is the number of zeroes (counted with multiplicity) of a holomorphic section of the line bundle. It is in fact true (see [9]) that every holomorphic line bundle over $\mathbb{C}P_1$ is isomorphic to some $\mathcal{O}(p)$. This means that if we have a holomorphic line bundle over $\mathbb{C}P_1$ that has a holomorphic section which has p zeroes then the line bundle has to be isomorphic to $\mathcal{O}(p)$. We shall use this result later.

In the case of $\mathcal{O}(p)$ defined on $\mathbb{C}P_n$ a similar result holds. If $p < 0$ then there are no sections and if $p > 0$ then the space of sections has the same dimension as the space of all homogenous polynomials in $n+1$ variables of degree p. To see why this space of polynomials arises recall that $\mathcal{O}(1) = H^*$ is the dual bundle of $\mathcal{O}(-1)$. We can define sections of $\mathcal{O}(1)$ by noting that for any $i = 1, \dots, n+1$ we can define a linear map on $\mathbb{C}^{n+1}$ and hence, by restriction on any line in $\mathbb{C}^{n+1}$, by picking out the ith component of a vector. Call this holomorphic section of $\mathcal{O}(1)$, ζ^i. Notice that it is non-vanishing on

U_i. These sections in fact span the space $H^0(\mathbb{C}P_n, \mathcal{O}(1))$ of all holomorphic sections. We can multiply or tensor these sections together p times to get a section of $\mathcal{O}(p)$ and so we find that $H^0(\mathbb{C}P_n, \mathcal{O}(-p))$ has the same dimension as the space of all homogeneous polynomials of degree p.

Example 5.5 (The section of $\mathcal{O}(k)$ defined by f). If we take the homogeneous polynomial $f(z)$ then $f(\zeta) = f(\zeta^0, \dots, \zeta^n)$ defines a section of $\mathcal{O}(k)$ vanishing precisely on X. This then is the solution to the problem raised in the previous section of extending the functions

$$f(\frac{z^0}{z^i}), \dots, 1, \dots, \frac{z^n}{z^i})$$

defined on the set U_i to all of $\mathbb{C}P_n$. We can but only if we intepret the result of doing so as a section of the line bundle $\mathcal{O}(k)$.

Example 5.6 (The bundle $\mathcal{O}(2)$ over Z). We have seen that S^2 and $\mathbb{C}P_1$ are the same as complex manifolds and that Z is the same real manifold as TS^2. It is also true that Z is isomorphic as a complex manifold, with the complex structure we have defined, to $T\mathbb{C}P_1$. Moreover, as we have noted, this is a line bundle on $\mathbb{C}P_1$. The complex holomorphic sections are just those described in section 2. Hence the space of all holomorphic sections of Z has the same dimension as the space of all homogenous polynomials of degree two in two variables. So it is a copy of $\mathbb{C}^3$. The real sections are a real subspace of this $\mathbb{R}^3$ as we discussed in section 2.

6 The canonical bundle of X

We want to integrate things over X. For this purpose we need to know what its canonical bundle is. It turns out that it is isomorphic to the bundle $\mathcal{O}(-n-1+k)$ restricted to X. I will sketch the proof of this result.

As X is a submanifold of $\mathbb{C}P_n$ the tangent space to X at any point of X is a subspace of the tangent space to all of $\mathbb{C}P_n$. The quotient of these is the normal space to X denoted by

$$N = T\mathbb{C}P_n/TX. \tag{10}$$

Notice that in Riemannian geometry we would use an inner product to actually pick out a normal. We have to resist the temptation to do this in complex geoemtry as it would involve using a hermitian inner product to pick out a normal subspace of $T\mathbb{C}P_n$ to X and that would involve taking conjugates and spoil the 'holomorphicness' of N. Note also that elementary linear algebra shows that $N^* \subset T\mathbb{C}P_n^*$, in fact it is the subspace of all linear functions on $T\mathbb{C}P_n$ which vanish on TX.

We have defined the canonical bundle of X to be the highest exterior power of TX^*. The highest exterior power of a vector space V is often denoted $\det(V)$ because if $X\colon V \to V$ is a linear map and V is, say, n dimensional then X induces a linear map from the one dimensional space $\bigwedge^n(V)$ to itself. Such a linear map must be multiplication by a complex number which is actually $\det(X)$. The quotient (10) induces a natural isomorphism $N \otimes \det(TX) = \det(T\mathbb{C}P_n)$. Note that N is one dimensional so that $N = \det(N)$. Hence if K is the canonical bundle of X we have

$$K = \det(TX)^* = \det(T\mathbb{C}P_n)^* \otimes N.$$

Now we only need to calculate N and the canonical bundle of $\mathbb{C}P_n$. This is actually a step forward!

Recall that f defines a holomorphic section of $\mathcal{O}(k)$ vanishing on X, $\mathcal{O}(k)$ has non-vanishing sections χ_i say over each of the open sets U_i and hence over these

$$f = f_i \chi_i$$

for some holomorphic functions $f_i\colon U_i \to \mathbb{C}$. On a complex manifold the usual exterior derivative

$$d = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} dx^i$$

has a holomorphic analogue

$$\partial = \sum_{i=1}^{n} \frac{\partial}{\partial z^i} dz^i.$$

Define

$$\partial f = (\partial f_i)\chi_i.$$

As f is constant on X this is zero when applied to vectors tangent to X and hence ∂f_i is a section of N^*. It follows that ∂f is a section, over U_i of

$$N^* \otimes \mathcal{O}(k).$$

It is, in fact, a non-vanishing section. The reason for this a little technical. If it vanished that would amount to f having a multiple root somewhere on X and that would contradict the assumption that X is a smooth submanifold of $\mathbb{C}P_n$. I do not wish to dwell on this point but refer the interested reader instead to [9].

Notice that if we change from U_i to U_j then $\chi_i = g_{ij}\chi_j$ and thus $f_i = g_{ji} f_j$ where g_{ij} is a non-vanishing holomorphic function. Differentiating gives

$$\partial f_i = \partial(g_{ij}) f_j + g_{ij} \partial f_j$$

and evaluating at any point on X where f_j vanishes gives

$$(\partial f_i)\chi_i = (\partial f_j)\chi_j$$

on X. Hence ∂f is a non-vanishing section of $N^* \otimes \mathcal{O}(k)$ over all of X. So we conclude that $N = \mathcal{O}(k)$.

Finally we calculate $\det(T\mathbb{C}P_n)$ or the canonical bundle of projective space. The tangent space to projective space at a line $[z]$ can be shown to be isomorphic to all the linear maps from the line $[z]$ to the quotient $\mathbb{C}^{n+1}/\mathcal{O}(-1)_{[z]}$. In other words

$$T\mathbb{C}P_n = (\mathcal{O}(-1)^* \otimes \mathbb{C}^{n+1})/\mathcal{O}(-1).$$

When we apply det to a tensor product it behaves like taking the determinant of a Kronecker product of matrices. The latter multiplies and the same is true for spaces. That is $\det(\mathbb{C}^m \otimes V) = \det(V)^{\otimes m}$ so

$$\det(T\mathbb{C}P_n)^* = \mathcal{O}(-n) \otimes \det(\mathbb{C}^{n+1})^* \otimes \mathcal{O}(-1) = \mathcal{O}(-n-1)$$

and finally

$$K = \mathcal{O}(-n-1+k).$$

We can now begin to explain the twistor correspondence we are interested in. For the time being we ignore the question of defining a twistor space. We use the basis $\zeta^0, \ldots, \zeta^n$ of $H^0(\mathbb{C}P_n, \mathcal{O}(1))$ to define an explicit isomorphism with $\mathbb{C}^{n+1}$ by

$$(z^0, \ldots, z^n) \mapsto \sum_{i=0}^{n} z^i \zeta^i.$$

It is possible to show that the restriction map

$$H^0(\mathbb{C}P_n, \mathcal{O}(1)) \to H^0(X, \mathcal{O}(1))$$

is an isomorphism but this needs some of the theory of sheaves and would divert us too far from the main task. The point is that sections of $\mathcal{O}(1)$ vanish on linear subspaces of $\mathbb{C}P_n$ whereas the subspace X is not linear becuase $k > 1$ hence we cannot have a section of $\mathcal{O}(1)$ vanishing on all of X. For further details look at [9]. Let us just note then that

$$\mathbb{C}^{n+1} = H^0(\mathbb{C}P_n, \mathcal{O}(1)) = H^0(X, \mathcal{O}(1)).$$

Choose some differential forms $\omega_0, \ldots, \omega_m$ on X of type $(0, n-1)$ where ω_l has values in $\mathcal{O}(-n-1+k-l)$. Then each

$$\omega_l(\sum_{i=0}^{n} z^i \zeta^i)^l$$

is a differential form with values in $\mathcal{O}(-n-1+k)$ and hence so is their sum. We can then integrate to define

$$\phi(z) = \int_X \sum_{i=0}^{m} \omega_l(\sum_{i=0}^{n} z^i \zeta^i)^l. \tag{11}$$

This is a function on $\mathbb{C}^{n+1}$. If we apply a monomial differential operator

$$\frac{\partial}{\partial z^{i_1}} \frac{\partial}{\partial z^{i_2}} \cdots \frac{\partial}{\partial z^{i_k}}$$

to $\phi(z)$ we obtain

$$\int_X \sum_{i=0}^{m} \omega_l l(l-1)\ldots(l-k+1)(\sum_{i=0}^{n} z^i \zeta^i)^{l-k} \zeta^{i_1} \zeta^{i_2} \ldots \zeta^{i_k}.$$

Hence if we apply D_f to $\phi(z)$ we obtain

$$\int_X \sum_{i=0}^{m} \omega_l l(l-1)\ldots(l-k+1)(\sum_{i=0}^{n} z^i \zeta^i)^{l-k} f(\zeta^0, \zeta^1, \ldots, \zeta^k).$$

which vanishes because f vanishes on X.

So far we have a transformation from collections of differential forms $\omega_0, \ldots, \omega_m$ to functions in the kernel of the differential operator D_f. Clearly an infinite sum of ω_l's would also work if the sum was convergent. We might also expect that sometimes the function ϕ may be the zero function. To understand these details we need to define a twistor space Z for this problem. Before doing that it is interesting to note that something similar to a Fourier transformation is going on here in as much as differentiation is being turned into multiplication. Eastwood has made this analogy precise by using the *twistor transform* [7].

7 Twistor space

To understand the construction introduced in the previous section we need to construct an analogue of mini-twistor space. We shall also call it Z in this case. We define Z to be the line bundle $\mathcal{O}(1)$ restricted to X. Notice that, in as much as X depends on f the twistor space Z also depends on f.

Example 7.1 (Minitwistors.). In the case of the Laplacian on $\mathbb{R}^3$ we have seen that $X = S^2 = \mathbb{C}P_1$ and sits inside $\mathbb{C}P_2$ as the subvariety

$$(z^0)^2 + (z^1)^2 + (z^2)^2 = 0.$$

via the map

$$w \mapsto [i(w^2+1), (w^2-1), 2w].$$

We have noted that Z is the line bundle $\mathcal{O}(2)$ over $\mathbb{C}P_1$ whereas I am now suggesting that it is the line bundle $\mathcal{O}(1)$ over $\mathbb{C}P_2$ restricted to X. This is not a contradiction because the line bundle $\mathcal{O}(1)$ on $\mathbb{C}P_2$ restricted to X and thought of as line bundle over $\mathbb{C}P_1$ is actually the line bundle $\mathcal{O}(2)$. To see this recall from the section 5 that we can find out which $\mathcal{O}(p)$ a line bundle over $\mathbb{C}P_1$ is by choosing a holomorphic section and counting its zeroes. In this case we can take a holomorphic section of $\mathcal{O}(1)$ given by the linear function z^0. This vanishes on the hyperplane $z^0 = 0$ and the restriction of this to the subvariety $X = \mathbb{C}P_1$ is the pair of points $[0, 1, i]$ and $[0, -1, i]$. Hence it vanishes at two places and so the line bundle in question is isomorphic to $\mathcal{O}(2)$.

The relationship between the twistor space Z and $\mathbb{R}^{n+1}$ revolves around the following *double fibration* where we identify $\mathbb{R}^{n+1}$ with a real subspace of $H^0(X, \mathcal{O}(1))$.

$$\begin{array}{ccc} & \mathbb{R}^{n+1} \times X & \\ \swarrow & & \searrow \\ \mathbb{R}^{n+1} & & Z \end{array} \tag{12}$$

If $(\sum_{i=0}^n x^i \zeta^i, x) \in \mathbb{R}^{n+1} \times X$ then the left hand map sends this to $x = (x^0, \ldots, x^n)$ and the right hand arrow sends it to $\sum_{i=0}^n \zeta^i(x)$. If we fix a point x in $\mathbb{R}^{n+1}$ then the image of the section

$$\sum_{i=0}^n x^i \zeta^i \colon X \to Z$$

is a subvariety X_z in Z which is is identified with X by the projection $\pi\colon Z \to X$. This is the analogue of the real section determined by the point x in the mini-twistor case.

Example 7.2 (Minitwistors). Recall from section 2 the construction of the mini-twistor space. We can introduce the double fibration

$$\begin{array}{ccc} & \mathbb{R}^3 \times S^2 & \\ \swarrow & & \searrow \\ \mathbb{R}^3 & & Z \end{array} \tag{13}$$

Here the left-hand arrow is the map $(x, u) \mapsto v$ and the right-hand arrow is the map $(v, u) \mapsto (v - < v, u > u, u)$ or in complex co-ordinates

$$(v, \zeta) \mapsto ((\frac{1}{2}((x + iy) + 2z\zeta - (x - iy)\zeta^2), \zeta).$$

We will need various line bundles on Z. We can use the projection $\pi \colon Z \to X$ to *pull-back* line bundles from X. That is, if L is a line bundle on X, we define a line bundle $\pi^{-1}L$ on Z by $(\pi^{-1}(L))_z = L_{\pi(z)}$. We adopt the convention of denoting $\pi^{-1}(\mathcal{O}(p))$ by just $\mathcal{O}(p)$. Notice that a peculiar thing happens for the bundle $\mathcal{O}(1)$. Here if z is an element of Z then because Z is itself the line bundle $\mathcal{O}(1)$ we have $z \in \mathcal{O}(1)_{\pi(z)}$ and hence $z \in \pi^{-1}(\mathcal{O}(1))_z$. We denote this section of $\mathcal{O}(1)$, $z \mapsto z$ by η. Notice that the variety X_z, in this notation, is the subset of Z where $\eta = \sum_{i=0}^n \zeta^i z^i$. Finally we remark that differential forms also pull back from X to Z. This is a standard piece of differential geometry, see for example [9, 10].

Recall the expression for ϕ in equation (11):

$$\phi(z) = \int_X \sum_{i=0}^m \omega_l (\sum_{i=0}^n z^i \zeta^i)^l.$$

where each of the ω_l is a $(0, n-1)$ form on X with values in $\mathcal{O}(-n-1+k-l)$ respectively. Consider the $(0, n)$ form on Z with values in $\mathcal{O}(-n-1+k)$ defined by

$$\omega = \sum_{i=1}^m \pi^{-1}(\omega_l)\eta^l.$$

Because each X_z is a copy of X we can restrict ω to X_z and integrate. We get

$$\begin{aligned} \int_{X_z} \omega &= \int_{X_z} \sum_{i=1}^m \pi^{-1}(\omega_l)\eta^l \\ &= \int_{X_z} \sum_{i=1}^m \pi^{-1}(\omega_l)(\sum_{i=0}^n z^i \zeta^i)^l \\ &= \int_X \sum_{i=1}^m \omega_l (\sum_{i=0}^n z^i \zeta^i)^l \end{aligned}$$

Hence

$$\phi(z) = \int_{X_z} \omega.$$

We can in fact define such a function ϕ by this formula for any differential form ω on the twistor space Z of type $(0, n-1)$ with values in $\mathcal{O}(-n-1+$

k). Some of these when integrated over any X_z give zero. The situation is analogous to that of a real manifold M of dimension n. In that case a differential n form can be integrated and an application of Stoke's theorem tells us that we will get zero if the differential form is of the form $d\mu$ for μ an $n-1$ form. The quotient of the space of all n forms by those of the form $d\mu$ is $H^n(M, \mathbb{R})$, the nth de-Rham cohomology group of M. In the complex situation the appropriate analogue of d is

$$\bar{\partial} = \sum_{i=1}^{n} \frac{\partial}{\partial \bar{z}^i} d\bar{z}^i.$$

which maps $(0, p)$ forms to $(0, p+1)$ forms. The space of all $(0, n-1)$ forms with values in $\mathcal{O}(-n-1+k)$ modulo those of the form $\bar{\partial}$ of something is denoted by

$$H^{n-1}(Z, \mathcal{O}(-n-1+k))$$

and called the $n-1$st Dolbeault cohomology of Z with values in $\mathcal{O}(-n-1+k)$. This is an infinite dimensional vector space (it would be finite dimensional if Z was compact). Finally we can define the twistor correspondence:

$$T\colon H^{n-1}(Z, \mathcal{O}(-n-1+k)) \to H^0(\mathbb{C}^{n+1}, \mathcal{O})$$

by $T(\omega)(z) = \int_{X_z} \omega$.

The theorem proved in [19] is that this map T is a bijection onto the kernel of D_f. The method of proof is to note that the expansion in terms of powers of η of a differential form ω corresponds to the power series expansion of $T(\omega)$. In algebraic geometry this expansion in powers of η corresponds to expanding the form ω normal to the subvariety X_0. It then remains to essentially compare terms in the power series.

8 The examples revisited

Let us revisit the examples in Whittaker and Watson [21]. The twistor theory tells us that on $\mathbb{R}^3$ a function in the kernel of the Laplacian is given by

$$\phi(x, y, z) = \int_{S^2} \sum g_l(w)(x\zeta^1 + y\zeta^2 + z\zeta^3)^l dw.$$

Restricting w to the circle and using a contour integral this becomes

$$\phi(x, y, z) = \int_{\pi}^{-\pi} \sum g_l(w) 2^l \exp(ilw)(xi\cos(w) + yi\sin(w) + z)^l dw.$$

Assuming the sum converges and defining

$$g(v, w) = \sum g_l(w) 2^l \exp(ilw) v^l$$

we recover the formula in section 3.

In the second case if the polynomial is

$$f(x, y, z, t) = x^2 + y^2 + z^2 - t^2$$

and X is again a quadric in $\mathbb{C}P_3$. In this case it is known that this quadric is a product of two spheres that is $X = \mathbb{C}P_1 \times \mathbb{C}P_1$. We can define 'co-ordinates' on this (actually a two to one map) by

$$(w, w') \mapsto [(1 + w^2)(w'^2 - 1), -i(w^2 - 1)(w'^2 - 1), 2i(1 + w'^2)^2, 4iww']$$

and a similar calculation yields the formula of Whittaker and Watson.

9 Correspondences

The double fibration in equation (12) is an example of a *correspondence*. More generally we say that a correspondence is a submanifold of $X \times Z$ such that the induced maps $W \to X$ and $W \to Y$ are fibrations. We summarise this data by considering the double fibration

$$\begin{array}{ccc} & W & \\ \swarrow & & \searrow \\ X & & Z \end{array} \tag{14}$$

Notice that a correspondence is a generalisation of the idea of having a map from X and Z or vice versa. For example if $F : X \to Z$ then the graph of F inside $X \times Z$ defines a correspondence. Similarly for a map from $Y \to X$.

Another way of obtaining a correspondence is to consider a Lie group G with two subgroups H and K. Then we have the diagram of homogeneous spaces:

$$\begin{array}{ccc} & G/(H \cap K) & \\ \swarrow & & \searrow \\ G/K & & G/H \end{array} \tag{15}$$

Because many representations arise as space of sections of bundles over homogeneous spaces we are naturally lead to Penrose transforms between representations. Many *integral intertwining operators* arise in this way [4].

The classical Radon transform also fits into this picture if we let $X = \mathbb{R}^3$, Z be the space of planes in $\mathbb{R}^3$ and W the subset of $\mathbb{R}^3 \times Z$ where the point in $\mathbb{R}^3$ lies on the plane [10].

Given a correspondence we can try and transform objects such as differential forms, sections of bundles and bundles backwards and forwards between Z and X via W. To do this we need to be able to pull objects back from Z and W and then push them down from W to X. A quite general machinery for this has been developed [6].

10 The Weierstrass representation

Examples of non-linear twistor transforms really go back to Weierstrass' work on minimal surfaces although Weierstrass, of course, was not thinking about twistors. Let me explain briefly how the classical Weierstrass representation of a minimal surface [17] fits into the context of mini-twistor space.

If Σ is an oriented surface in $\mathbb{R}^3$ then any point p of Z there is a well-defined oriented normal line ℓ_p. Hence we have a map

$$\begin{array}{ccc} \Sigma & \to & Z \\ p & \mapsto & \ell_p \end{array}$$

whose image defines a submanifold $\widetilde{\Sigma}$ of Z. Weierstrass' result was that if the surface Σ is minimal then $\widetilde{\Sigma}$ is a holomorphic submanifold of Z. There is also a reverse correspondence that constructs a minimal surface out of a holomomorphic submanifold of Z.

Note that if we compose with the projection from Z to S^2 that maps a line to the unit normal in its direction then we obtain the classical Gauss map which associates to a point on the surface its unit normal in S^2. For further details on the Weierstrass representation from this point the reader should look at the paper of Hitchin [11].

11 Monopoles

The original work in this area was the Atiyah-Ward transform for the self-duality equations [3] on $\mathbb{R}^4$. A good reference are the Pisa lecture notes of Atiyah [1]. These can be hard to find so it is worth noting that they are in his collected works. As I am more interested in monopoles and they fit into the mini-twistor space setting we have developed here let me say something briefly about them. They are in any case the time invariant version of the self-duality equations. The best place to learn about monopoles is the book by Atiyah and Hitchin [2] and the references therein.

Monopoles are a guage theory on $\mathbb{R}^3$. To define a monopole we start with a pair (A, ϕ) consisting of a connection 1-form A on $\mathbf{R}^3$ with values in

$LSU(2)$, the Lie algebra of $SU(2)$, and a function ϕ (the Higgs field) from $\mathbf{R}^3$ into $LSU(2)$. The Yang-Mills-Higgs energy on this pair is

$$\mathcal{E}(A,\phi) = \int_{\mathbf{R^3}} (|F_A|^2 + |\nabla_A \phi|^2) d^3x$$

where $F_A = dA + A \wedge A$ is the curvature of A, $\nabla_A\phi = d\phi + [A,\phi]$ is the covariant derivative of the Higgs field, and we use the usual norms on 1-forms and 2-forms and the standard inner product on $LSU(2)$. The energy is minimized by the solutions of the Bogomolny equations

$$\star F_A = \nabla_A \phi \tag{16}$$

where $\star$ is the Hodge star on forms on $\mathbf{R}^3$. These equations, and the energy, are invariant under gauge transformations, where the gauge group $\mathcal{G}$ of all maps g from $\mathbf{R}^3$ to $SU(2)$ acts by

$$(A,\phi) \mapsto (gAg^{-1} - dgg^{-1}\,,\, g\phi g^{-1}).$$

Finiteness of the energy, and the Bogomolny equations, imply certain boundary conditions at infinity in $\mathbf{R}^3$ on the pair (A,ϕ) which are spelt out in detail in [2]. In particular, $|\phi| \to c$ for some constant c which is usually take to be 1.

A monopole, then, is a gauge equivalence class of solutions to the Bogomolny equations subject to these boundary conditions. In some suitable gauge there is a well-defined Higgs field at infinity

$$\phi^\infty \colon S^2_\infty \to S^2 \subset LSU(2)$$

going from the two sphere of all oriented lines through the origin in $\mathbb{R}^3$ to the unit two-sphere in $LSU(2)$. The degree of ϕ^∞ is a positive integer k called the magnetic charge of the monopole.

Because the Bogomolny equations are non-linear the set of all solutions is not a linear space. After we quotient it by the gauge group it is a manifold M_k of dimension $4k$ called the moduli space of charge k monopoles. In the case that $k = 1$ there is a spherically symmetric monopole called the Bogomolny-Prasad-Sommerfield (BPS) monopole, or unit charge monopole. Its Higgs field has a single zero at the origin, and its energy density is peaked there so it is reasonable to think of the origin as the centre or location of the monopole. The Bogomolny equations are translation invariant so this monopole can be translated about $\mathbb{R}^3$ and also rotated by the circle of constant diagonal gauge transformations. This in fact generates all of M_1 which is therefore diffeomorphic to $S^1 \times \mathbb{R}^3$. The coordinates on M_1 specify the location of the monopole and what can be thought of as an internal phase. More generally there is an asymptotic region of the moduli space M_k consisting of approximate superpositions of k unit charge monopoles located at k widely separated points and with k arbitrary phases.

Hitchin [11, 12] developed the Atiyah-Ward correspondence for monopoles and this transforms a monopole into a holomorphic vector bundle on Z. The vector bundle is easy to describe. If γ is an oriented line we can restrict the connection and Higg's field to the line and consider the ordinary one variable differential equation

$$(\nabla_{\dot{\gamma}} - \Phi)\psi = 0. \tag{17}$$

The space of solutions to this a two dimensional complex vector space E_γ depending on γ. As such it forms a vector bundle $E \to Z$. The real trick now is to make this a *holomorphic* vector bundle. To do this the original connection and Higg's field have to satisfy the the Bogomolny equations. Conversely given this bundle we can recover the monopole and the solution of the Bogomolny equation.

Hitchin uses this Atiyah-Ward formalism to prove a number of interesting results. Let me conclude by mentioning the *spectral curve* of a monopole. Generally the equation (17) has no solutions that decay at both ends of the line. The set of all lines for which it does have such solutions form a special subset of Z called the *spectral curve*. This is determined by an equation of the form

$$\eta^k + a_1(\zeta)\eta^{k-1} + \cdots + \alpha_k(\zeta) = 0$$

where each of the a_i is a polynomial of degree $2i$. So, in particular, the spectral curve is holomorphic. This sort of result would be enormously difficult to prove with just analysis in $\mathbb{R}^3$. Hitchin goes on to show how to recover the monopole from its spectral curve [11] and to show precisely which curves correspond to monopoles [12]. This is just the beginning of an exciting story. We refer the reader to the book of Atiyah and Hitchin [2] and references therein for the details.

References

[1] M.F. Atiyah: *Geometry of Yang-Mills Fields*, Lezione Fermione, Pisa 1979. (Also appears in Atiyah's collected works.)

[2] M.F. Atiyah and N.J. Hitchin: *The geometry and dynamics of magnetic monopoles* Princeton University Press, Princeton, 1988.

[3] M.F. Atiyah and R.S Ward: *Instantons and algebraic geometry*, Commun. Math. Phys. **55**, (1977), 117-124. (Also appears in Atiyah's collected works.)

[4] R.J. Baston and M.G. Eastwood: *The Penrose Transform, its interaction with representation theory.* Oxford University Press, Oxford, 1989.

[5] M.G. Eastwood and R.O. Wells. *Cohomology and massless fields.* Commun. Math. Phys. **78**, (1981), 305–351.

[6] M.G. Eastwood: *The generalised Penrose-Ward transform*. Math. Proc. Camb. Phil. Soc. (1985), **97**, 165–187.

[7] M.G. Eastwood: *On Michael Murray's twistor correspondence* in [15].

[8] M.G. Eastwood: *Introduction to the Penrose Transform*, Contemporary Mathematics, **154**, 1993, 71–75.

[9] P. Griffiths and J. Harris: *Principles of algebraic geometry*, Wiley, 1978, New York.

[10] S. Helgason: *Differential geometry, Lie groups and symmetric spaces.* Academic Press, New York, 1978.

[11] N.J. Hitchin: *Monopoles and geodesics*, Commun. Math. Phys. **83** 579–602, 1982.

[12] N.J. Hitchin: *On the construction of monopoles*, Commun. Math. Phys. **89** 145–190, 1983.

[13] N.S. Manton, N.J. Hitchin and M.K. Murray: *Symmetric monopoles.* Preprint 1995.

[14] *Advances in Twistor Theory*, (eds. L.P. Hughston and R.S. Ward), Pitman Research Notes in Math. **37** (1979).

[15] *Further Advances in Twistor Theory*, (eds. L.J. Mason and L.P. Hughston), Pitman Research Notes in Math. **231** (1990).

[16] *Further Advances in Twistor Theory Volume II*, (eds. L.J. Mason, L.P. Hughston and P.Z. Kobak), Pitman Research Notes in Math. **232**, (1995).

[17] J. Lucas, M. Barbosa, and A. Gervasio Colares: *Minimal surfaces in* $\mathbb{R}^3$*.* Lecture Notes in Mathematics **1195**, Springer-Verlag, New York 1986.

[18] J.R. Munkres: *Topology a first course.* Prentice-Hall, New Jersey, 1975.

[19] M.K. Murray *A twistor correspondence for homogeneous polynomial differential operators.* Math. Ann. **272** , 99–115 (1985).

[20] R.O. Wells: *Differential Analysis on Complex Manifolds* Springer-Verlag, 1973, New York.

[21] E.T. Whittaker and G.N. Watson, *A course in modern analysis*, Cambridge, Cambridge University Press, 1950.