FREE FIELDS ON RIEMANN SURFACES AND SPECTRAL CURVES OF THE CHIRAL POTTS MODEL.

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Abstract. We report on progress towards an attempt to understand the representations of infinite dimensional groups that arise from quantum field theories on Riemann surfaces. We consider some examples that arise from the correspondence between chiral Potts integrable models and monopoles established by Atiyah and Murray and explore the Fermi-Bose correspondence on a general Riemann surface.

Mathematics Subject Classification. 22E67, 30F99, 81T30, 81T40,

1. Introduction.

Although there has been considerable discussion of quantum field theory on general Riemann surfaces, [AG], [E], [N], these accounts have been both predominantly in terms of local co-ordinates and purely algebraic (i.e. lacking the analytic information enabling a global treatment of the representations of the infinite dimensional groups associated with the surface analogous to the case of the Riemann sphere or disc [S], [PS], [CH], [CR] and the torus or annulus [CH1], [CH2]). Motivated by applications we have uncovered further examples which indicate a rich geometric structure (some of this is described in the work on hyperelliptic curves in [CEH]). Our work is directed towards understanding this geometry and the representations of infinite dimensional groups which one would expect, on the basis of the theory for the disc and annulus, to be associated with higher genus surfaces and in the variety of ways in which these come up particularly in connection with statistical mechanics. (The latter connections having to do with conformal invariance of statistical mechanical models.) Initial steps in this direction were taken by us (for fermions) in [CH] and by Jaffe et al (for bosons) in [JKL]. The present account verifies many of the conjectures made in [CH] and considerably extends the results of [JKL]. The complete details would take rather more space than we have here and so this progress report will:

(i) explore some interesting examples suggested by the chiral Potts-monopole connection established by Atiyah and Murray [A2],

(ii) demonstrate the geometric, co-ordinate free nature of free bosons and fermions on a Riemann surface,

(iii) sketch the analytic ideas needed for a global treatment of the Fermi-Bose correspondence on a general Riemann surface Σ.

Of course our thinking has been greatly influenced by ideas of Segal [S1] and in some sense this work should be seen in the context of exploring the vast range of examples which fit into that framework. Full proofs of our results will be published elsewhere. We begin the discussion with (iii) concentrating on how our approach differs from the usual one. The strategy we employ for handling the fermion-boson correspondence is to start with a fermionic Hilbert space, $\mathcal{K}$, associated geometrically with the surface. The CAR algebra of $\mathcal{K}$ is then generated by elements $\Psi(\phi)^*$, $\Psi(\psi)$ for ϕ and ψ in $\mathcal{K}$, subject to the anticommutation relations

$$\Psi(\phi)^*\Psi(\psi) + \Psi(\psi)\Psi(\phi)^* =< \phi, \psi > 1.$$

One then constructs the Fock representation associated with some geometrically defined projection on $\mathcal{K}$. The details of this are contained in section 2.

The bosons are regarded initially as gauge transformations of the fermion fields, and described by a group $\mathcal{G}$ which has a representation on $\mathcal{K}$. The first step towards establishing the Fermi-Bose correspondence is then to show that the transformations in $\mathcal{G}$ are implemented in the Fock representation space $\mathcal{F}$ of the fermions. That is, there exists a 2-cocycle σ for $\mathcal{G}$ and a σ-representation Γ of $\mathcal{G}$ on $\mathcal{F}$ such that

$$\Gamma(g)\Psi(\phi)\Gamma(g)^{-1} = \Psi(g \cdot \phi),$$

for all g in $\mathcal{G}$ and ϕ in $\mathcal{K}$. The projective or σ-representation Γ can then be interpreted as giving the boson commutation relations in Weyl form, for which the fermion vacuum vector turns out to define a boson Fock vacuum also. This gives the first part of the correspondence and is described in detail in section 5.

The converse path from bosons to fermions is more delicate. The case in which the analysis is most complete is where the surface Σ is a Schottky double of a surface Σ_1 with boundary a union of parametrised circles, $S_1, \ldots, S_n$. We have established the existence of families of elements

$$\{\gamma_{\lambda,a} \mid a \in \partial\Sigma_1, \lambda \in \Sigma_1\}$$

in $\mathcal{G}$ such that as λ converges to a point on a boundary component of Σ_1, a suitable multiple of $\Gamma(\gamma_{\lambda,a})$ 'converges to a fermion operator $\tilde{\Psi}_a$ at a'. More precisely this is short hand for the fact that the operator

$$\tilde{\Psi}(\phi) = \int \phi(a)\tilde{\Psi}_a \, da$$

is the limit (in an appropriate sense) of a suitable multiple of

$$\int \phi(a)\Gamma(\gamma_{\lambda,a})da$$

where the integral is over the boundary circle containing a, ϕ varies over an appropriate space of test functions and the operator $\tilde{\Psi}(\phi)$ lies in the fermion algebra over L^2 of the boundary circle. One of the interesting features of this viewpoint is that the functions $\gamma_{\lambda,a}$ are meromorphic on Σ with the minimum number of poles and zeros and have modulus one on the boundary $\partial\Sigma_1$.

Our concern with this mathematical view of the correspondence stems from the well-known technical difficulties which beset quantum field theory; the limiting procedure above reproduces the suggestive 'exponential of a free quantum field' (if viewed the right way) but, having precise analytical content, it may be used to prove facts about the representations of the infinite dimensional groups which arise in this context (cf. [S], [CR], [CH]). The analogues for general Riemann surfaces of Segal's vertex operators for the disc are clearly the operators $\Gamma(\gamma_{\lambda,a})$. We think of this as a kind of 'smoothed out' vertex operator and indeed this is useful in giving meaning to the vertex operators or insertions which are used in conformal field theory.

In fact, the above procedure contains some redundancy because the reconstruction of the fermions from the bosons can itself be regarded as an explicit construction of a representation of the CAR algebra. Since the GNS construction determines the representation uniquely from its correlation functions, and it is then possible to find explicitly how the boson operators are implemented in this form so that the derivation of bosons from fermions can be deduced from the description of the fermions in terms of the bosons. This latter viewpoint forms the basis of our strategy of proof however we defer the details to a later publication as they are quite lengthy. Our aim in the rest of this paper is to give our geometric view of the theory and some interesting new examples.

We begin with the way in which $\mathcal{K}$ and $\mathcal{G}$ are derived from the geometry, and describe the kernels which determine the projections defining the Fock states.

2. Fermions on a Riemann surface

Given a Riemann surface Σ, a covering by two proper open sets U_1 and U_2 and a line bundle L over Σ we denote by $\Gamma(\Sigma, \mathcal{O}(L))$ the sheaf of germs of holomorphic sections of L and by $H^1(\Sigma, \mathcal{O}(L))$ the first cohomology group with coefficients in this sheaf. The Mayer-Vietoris sequence gives

$$0 \to \Gamma(\Sigma, \mathcal{O}(L)) \to \Gamma(U_1, \mathcal{O}(L)) \oplus \Gamma(U_2, \mathcal{O}(L)) \to \Gamma(U_1 \cap U_2, \mathcal{O}(L)) \to H^1(\Sigma, \mathcal{O}(L)) \to 0.$$

This sequence gives information in a variety of cases although we will discuss only two: namely where L is the canonical bundle K (the bundle of holomorphic one forms) and the case where L is a spin structure (i.e. a square root of K). So let $K^{\frac{1}{2}}$ be an even spin structure for which $\Gamma(\Sigma, \mathcal{O}(L))$ vanishes (this is the generic case [F]). Serre duality forces the last term of the sequence to also vanish and so the space of holomorphic sections of $L|_{U_1 \cap U_2}$ splits into a direct sum of sections holomorphic over U_1 and U_2 respectively.

Now suppose U_1 and U_2 are neighbourhoods of closed submanifolds Σ_1 and Σ_2 with smooth boundary such that

$$\Sigma_1 \cap \Sigma_2 = \partial\Sigma_1 = \partial\Sigma_2.$$

Choose an orientation of the boundary and regard this as a path in Σ. Then given sections $\alpha_i, i = 1, 2$ of $\Gamma(U_1 \cap U_2, \mathcal{O}(K^{\frac{1}{2}}))$ we have $\alpha_1 \otimes \alpha_2$ a section of K so we may integrate this over $\partial\Sigma_1$ to get a natural bilinear form on the holomorphic sections of $K^{\frac{1}{2}}$ restricted to $U_1 \cap U_2$:

$$(\alpha_1, \alpha_2) = \int_{\partial\Sigma_1} \alpha_1 \alpha_2.$$

The Clifford algebra of this space will play an important role in what follows.

> **Remark 2.1** If α_1 and α_2 are both in the image of $\Gamma(U_1, \mathcal{O}(K^{\frac{1}{2}}))$ under the Mayer-Vietoris map then their product has a holomorphic extension to Σ_1 and so by Cauchy's Theorem $(\alpha_1, \alpha_2) = \int \alpha_1\alpha_2$ vanishes. Thus $\Gamma(U_1, \mathcal{O}(K^{\frac{1}{2}}))$ is isotropic, and the same applies to $\Gamma(U_2, \mathcal{O}(K^{\frac{1}{2}}))$.

Denote by K_j the spaces $\Gamma(U_j, \mathcal{O}(K^{\frac{1}{2}}))$ for $j = 1, 2$. By Remark 2.1 both these subspaces are isotropic and so the Mayer-Vietoris sequence provides a decomposition

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$$

into isotropic subspaces.

Whenever one has such an isotropic decomposition of a space with bilinear form it is well known that there is a natural representation Ψ of the Clifford algebra of $\mathcal{K}$ on the exterior algebra $\wedge\mathcal{K}_1$. An element $h \in \mathcal{K}_1$ is represented by exterior multiplication, whilst an element $h \in \mathcal{K}_2$ acts by inner multiplication

$$\Psi(g)\Psi(h) + \Psi(h)\Psi(g) = (g, h)1,$$

for all g and h in $\mathcal{K}_1 \oplus \mathcal{K}_2$.

The cyclic vector $\Omega_1 = 1 \oplus 0 \oplus 0 \ldots \in \wedge\mathcal{K}_1$ is called the vacuum vector. There is a similar vector Ω_2 in $\wedge\mathcal{K}_2$. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are isotropic the bilinear form provides a pairing between them, which can be extended to their exterior algebras. Writing P_1 for the (not necessarily orthogonal) projection onto $\mathcal{K}_1$ along $\mathcal{K}_2$ we then readily check from the definitions that

$$(\Omega_2, \Psi(g)\Psi(h)\Omega_1) = (g, P_1 h).$$

Note that P_1 and the projection P_2 onto $\mathcal{K}_2$ are transpose maps with respect to the bilinear form.

The case of the hyperelliptic curves studied in [CEH] shows that interesting results may be established and applied even in this generality. Note that the discussion above includes the Krichever-Segal-Wilson example [SW].

3. Schottky doubles

In the more conventional physics literature $\mathcal{K}_1$ has a pre-Hilbert space structure and the Clifford algebra is replaced by the CAR algebra. Both these desiderata can be attained by working with a Schottky double. For this we take the space Σ_2 to be an oppositely oriented copy of Σ_1 with the boundaries identified in the obvious way. The closed surface $\Sigma = \Sigma_1 \cup \Sigma_2$ is then called the Schottky double of Σ_1, [F], [H]. There is a natural (antiholomorphic) involution ϕ which interchanges a

point z of Σ_1 and the corresponding point $\tilde{z}$ of Σ_2, so that $\tilde{z} = \phi(z)$ and $z = \phi(\tilde{z})$. By definition ϕ fixes the boundary $\partial\Sigma_1$.

This geometric involution induces an antilinear involution on the holomorphic $\frac{1}{2}$-forms defined by

$$\tilde{\alpha}(z) = \overline{\alpha(\tilde{z})},$$

where $\alpha(\tilde{z})$ is shorthand for the action of the map induced by ϕ on the half-form α. This involution maps $\mathcal{K}_1$ to $\mathcal{K}_2$ and vice versa. Now it is clear that (since ϕ fixes the points of $\partial\Sigma_1$)

$$< \alpha_1, \alpha_2 >= (\tilde{\alpha_1}, \alpha_2) = \int_{\partial\Sigma_1} \overline{\alpha_1(z)}\alpha_2(z)$$

defines a Hilbert space inner product. Henceforth let $\mathcal{K}$ denote the completion of

$$\Gamma(\partial\Sigma_1, \mathcal{O}(K^{\frac{1}{2}}))$$

in this Hilbert space topology. By taking a sequence of neighbourhoods U_1 and U_2 which shrink down to Σ_1 and Σ_2 respectively we obtain two increasing sequences of subspaces of the space $\mathcal{K}$ namely the spaces $\Gamma(U_j, \mathcal{O}(K^{\frac{1}{2}}))$ for $j = 1, 2$. For convenience we shall write the respective closures of these increasing unions as $\mathcal{K}_j$. By Remark 2.1 both these subspaces are isotropic. As U_1 and U_2 shrink the sequence $\Gamma(U_1 \cap U_2, \mathcal{O}(K^{\frac{1}{2}}))$ increases to a dense subspace of $\mathcal{K}$. Hence the Mayer-Vietoris sequence provides a decomposition

$$\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$$

into isotropic subspaces. This enables us to regard both $\mathcal{K}_1$ and $\wedge\mathcal{K}_1$ as Hilbert spaces, and it is simple to show that in this case P_1 is an orthogonal projection.

It is also easy to check that for any $\xi \in \wedge\mathcal{K}_1$ we have

$$< \Psi(f)\Omega_1, \xi >=< f, \xi >= (\tilde{f}, \xi) = (\Omega_2, \Psi(\tilde{f})\xi).$$

¿From this it immediately follows that

$$< \Omega_1, \Psi(f)^*\xi >= (\Omega_2, \Psi(\tilde{f})\xi),$$

and, in particular, that

$$< \Omega_1, \Psi(f)^*\Psi(g)\Omega_1 >= (\Omega_2, \Psi(\tilde{f})\Psi(g)\Omega_1) = (\tilde{f}, P_1 g) =< f, P_1 g > .$$

Similar calculations show that for any f and g in $\mathcal{K}_1$ the Clifford algebra relation become those of the CAR algebra:

$$\Psi(f)^*\Psi(g) + \Psi(g)\Psi(f)^* =< f, g > 1,$$

and

$$\Psi(f)\Psi(g) + \Psi(g)\Psi(f) = 0.$$

The formula for the two point correlation function derived in the preceding

paragraph is the usual one for Fock space, and the same representation could have been constructed by the GNS theory.

Another key tool is the Szegö kernel, $\Lambda = \Lambda_\Sigma$, which is the integral kernel of the projection operator P_1; that is for any $\alpha \in \mathcal{K}$

$$P_1\alpha(x) = \int_{\partial\Sigma_1} \overline{\Lambda(\tilde{x}, y)}\alpha(y) = \int_{\partial\Sigma_1} \Lambda(\tilde{y}, x)\alpha(y).$$

Since ϕ fixes the points of $\partial\Sigma_1$ this may also be written as

$$P_1\alpha(x) = \int_{\partial\Sigma_1} \Lambda(y, x)\alpha(y).$$

We can therefore write the formula for a general correlation function in the form

$$(\alpha, P_1\beta) = \int_{\partial\Sigma_1} \Lambda(y, x)\alpha(x)\beta(y).$$

The fact that P_1 and P_2 are transpose projections means that $\Lambda(x, y)$ is the kernel for P_2, and from the fact that the projections are complementary we see that

$$\Lambda(y, x) + \Lambda(x, y) = \delta(x, y)\sqrt{dxdy}.$$

This formula is familiar in Riemann surface theory [F] in the form that

$$\Lambda(y,x)+\Lambda(x,y)=0$$

when x and y are in the interior of Σ_1. It can also be deduced by a careful use of Cauchy's Theorem.

The Szegö kernel can be written explicitly in terms of the theta function

$\theta[e]$ associated to the same even half-period e which specifies the

choice of spin bundle L, and the Schottky-Klein prime form E, which is a $-\frac{1}{2}$-form in each of its arguments. Explicitly we have

$$\Lambda(x,y)=\frac{\theta[e](y-x)}{2\pi i\theta[e](0)E(y,x)}.$$

This formula makes it clear that Λ can be defined for any surface Σ whether or not it is a Schottky double. A different but not unrelated view of these correlations is contained in [R] and references therein.

The payoff for our non-Hilbert space approach comes about as follows. It is known on general grounds that all Riemann surfaces having the same boundary $\partial\Sigma_1$ define equivalent Hilbert space representations of the CAR algebra. This fact is an extension of [PS Section 8.11] which we will not digress to establish here. This means that we may relate the two-point functions for different 'cappings' of Σ_1 by surfaces Σ_2. In other words we may obtain a formula for the two point function associated with any 'cap' Σ_2 in terms of Szego kernels and intertwining operators. The precise details are notationally messy so we defer them to a more detailed exposition. The conclusion however to this section is that we now have a way of realising the functorial approach of Segal [S] in terms of correlation functions at least for free fermions.

4. Monopoles and the chiral Potts' examples

In this section we show how the monopole view of the chiral Potts spectral curves gives rise to a Schottky double fermionic theory of the type described in the previous section. Recall firstly that monopoles in hyperbolic space [A1] are determined by certain algebraic curves in the minitwistor space associated to hyperbolic space which is the quadric surface $Q=\mathbf{P}_1\times\mathbf{P}_1$.

Recall that holomorphic line bundles on $\mathbf{P}_1$ are uniquely determined by their first Chern class which is an integer. For each integer k there is the holomorphic line bundle denoted by $\mathcal{O}(k)$. Similarly on the quadric $\mathbf{P}_1\times\mathbf{P}_1$ holomorphic line bundles are also determined uniquely by their first Chern class in $H^2(\mathbf{P}_1\times\mathbf{P}_1,Z)=Z\oplus Z$. They are therefore determined by two integers m and n and we denote the corresponding line bundle by $\mathcal{O}(m,n)$. To be completely precise there are the two projection maps $\pi_i:Q\to\mathbf{P}_1$, for $i=1,2$ onto each of the two factors. Then we have

$$\mathcal{O}(m,n)=\pi_1^*(\mathcal{O}(m))\otimes\pi_2^*(\mathcal{O}(n)).$$

the line bundle obtained by first pulling back $\mathcal{O}(m)$ from the first factor then $\mathcal{O}(n)$ from the second factor and then tensoring them together. Its cohomology can be calculated by a Kunneth formulae from the cohomology of the two factors. Let us denote by L the line bundle $\mathcal{O}(1,-1)$.

A monopole in hyperbolic space has two important parameters, p and N. Usually N is called k but we need k for something else in the chiral Potts' model. The magnetic charge N is an integer, the winding number of the Higgs field at infinity. The norm of the Higgs field at infinity p is a non-negative real number. Every monopole is an S^1 invariant instanton on the product of hyperbolic space with the circle. This four manifold is conformally just the S^4 minus a two sphere. The monopoles which extend to instantons on all of the four sphere are those with integral p as discussed in [A1]. This discussion is concerned with these monopoles.

The monopole is determined in the twistor picture by a 'spectral' curve S which is the divisor of a section of $\mathcal{O}(N,N)$. This curve is special for various reasons, in particular the line bundle $L^{2p+N} = \mathcal{O}(-2p-N, 2p+N)$ is trivial over it. In the case that $p = 0$ Atiyah and Murray showed that these curves are those that arise in the N state chiral Potts' model [A2]. Then you have a curve with $\mathcal{O}(-N,N)$ trivial over it. The genus of the curve in either case is $(N-1)^2$. So, for instance, if $N = 1$ these curves are spheres and if $N = 2$ they are elliptic curves.

The canonical bundle of the quadric is $\mathcal{O}(-2,-2)$, i.e. the tensor product of the canonical bundles of the two factors. As the normal bundle to the curve S is $\mathcal{O}(N,N)$ it is straightforward as in Griffiths and Harris [GH] to see that the canonical bundle of the curve is $K = \mathcal{O}(N-2, N-2)$. Note that this is correct for $N = 1, 2$. A spin structure then is a square root of this. We shall choose $\mathcal{O}(-1, N-1)$ as a spin structure. Notice that this is not really asymmetrical in the two factors as over S one has

$$\mathcal{O}(N-1,-1) = \mathcal{O}(N-1,-1) \otimes \mathcal{O}(-N,N) = \mathcal{O}(-1,N-1).$$

If we square this we get over S

$$\mathcal{O}(2N-2,-2) = \mathcal{O}(2N-2,-2) \otimes \mathcal{O}(-N,N) = \mathcal{O}(N-2,N-2).$$

It is automatic from the Kunneth fumulae and the fact that $\mathcal{O}(-1)$ has no cohomology that $\mathcal{O}(N-1,-1)$ has no cohomology as a bundle on the quadric. To see how it behaves on the curve we use the exact sequence defining the structure sheaf of the curve

$$0 \to \mathcal{O}_Q(-1,-N-1) \to \mathcal{O}_Q(N-1,-1) \to \mathcal{O}_S(N-1,-1) \to 0.$$

This induces a long exact sequence in sheaf cohomology but the first two sheaves have no cohomology over the quadric so the last sheaf has no cohomology.

We now have most of the ingredients to discuss fermions on the spectral curve. The final ingredient is to divide the spectral curve into two halves. We do this by noting that this curve is identified with the set of (complex) Boltzmann weights of the N state chiral

Potts' model. The real Boltzmann weights form a subset of the curve that we will show is a collection of circles dividing it into two. The curve is defined by the equation

$$k((-1)^N\eta^N\zeta^N+1)-((-1)^N\eta^N+\zeta^N)=0$$

where η and ζ are co-ordinates on the two $\mathbf{P}_1$'s and $0<k<1$. In the chiral Potts models the real Boltzmann weights occur when both these co-ordinates are on the circle. Consider the map

$$(\eta,\zeta)\mapsto(1/\bar{\eta},1/\bar{\zeta}).$$

This is an anti-holomorphic involution of the quadric whose fixed points are precisely the product of the equators of the two spheres in the quadric. Note that this involution fixes the curve and that the real Boltzmann weights are the fixed points of the involution acting on the curve. We can write the equation of the curve as

$$\eta^N=\frac{-1}{(-1)^N}\frac{\zeta^N-k}{1-\zeta^N k}.$$

Consider the fractional linear transformation

$$w=\frac{-1}{(-1)^N}\frac{(v-k)}{(1-kv)}=\frac{-1}{(-1)^N v}\frac{(v-k)}{(v^{-1}-k)}.$$

Notice that if $|v|=1$ then $v^{-1}=\bar{v}$ so that the quotient in this expression is of the form complex number divided by its conjugate ie it is a phase. Hence the whole expression is on the circle. So if ζ is on the circle so also is η. Moreover if $v=0$ then $w=(-1)^N k$ so this fractional linear transformation maps the inside of the circle to the inside of the circle. It follows that the the curve we are considering is a union of two submanifolds with boundary

$$\Sigma_1=\{|\eta|,|\zeta|>1\}$$

and

$$\Sigma_2=\{|\eta|,|\zeta|<1\}$$

whose common boundary is

$$C=\{|\eta|,|\zeta|=1\}.$$

We claim that C is a union of N circles. To show this it is enough to know that the function

$$g_k(\theta)=\frac{-1}{(-1)^N}\frac{e^{iN\theta}-k}{1-e^{iN\theta}k}.$$

has winding number N as then the function f defined by

$$e^{iNf(\theta)}=\frac{-1}{(-1)^N}\frac{e^{iN\theta}-k}{1-e^{iN\theta}k}.$$

is periodic and the N circles are parametrised by $\zeta = \omega e^{i\theta}$, $\eta = e^{if(\theta)}$ where ω runs over the Nth roots of unity. But the winding number of g_k is independent of k as g_k varies continously with k and at $k = 0$ we have

$$g_0 = \frac{-1}{(-1)^N} e^{iN\theta}.$$

which has winding number N.

Notice that the involution flips Σ_1 and Σ_2 and that the chiral Potts' model curve is an example of a Schottky double. We note that the case $N = 2$ is the Ising model and even in that example the meaning of this fermionic theory is not known. It is nevertheless tempting to speculate on the existence of a 'Grassmannian picture' for these lattice models analogous to that which holds for integrable non-linear equations (cf [PS] and references therein).

5. Bosons and gauge transformations on a Riemann surface

There are, broadly speaking, two different approaches to the bosons. One can, in the spirit of loop groups, regard them as arising from the action of the gauge transformation group $\mathcal{G} = \mathrm{Map}(\partial\Sigma_1, \mathbf{T})$ by pointwise multiplication on the fermion space $\mathcal{K}$. We have already noted that the fermion representation, being equivalent to that obtained by simply capping each boundary circle with a disc, is equivalent to the tensor product of representations of fermions on the individual boundary components. The boson action can therefore be implemented by a projective representation Γ of $\mathcal{G}$ equivalent to the tensor product of those for the loop groups on individual boundary circles:

$$\Gamma(\xi)\Psi(\alpha)\Gamma(\xi)^{-1} = \Psi(\xi \cdot \alpha).$$

The cocycle σ for the representation is, of course the product of the standard cocycle for the different boundary circles:

$$\sigma(e^{if}, e^{ig}) = \exp[-\frac{i}{4\pi}(\int_{\partial\Sigma_1} gdf + \sum g(c_j)\Delta_j(f))],$$

where c_j is a fixed point on the j-th boundary circle $\partial_j\Sigma_1$ and $\Delta_j(f)$ is the difference in $f(z)$ as z traverses that boundary circle once. We shall give an alternative geometric description of the cocycle in the next section.

The second approach to the boson theory looks rather at the level of Lie algebras and uses meromorphic forms on the Riemann surface, see, for example, [AG], [E], [JKL] and [N]. This can be linked to the loop group approach in the following way.

Let $\mathcal{C}$ be the subgroup of local constants in $\mathcal{G}$, that is the $\mathbf{T}$-valued maps on $\partial\Sigma_1$ which are constant on each boundary component. Thus $\mathcal{C} \cong \mathbf{T}^{p+1}$ where $p + 1$ is the number of boundary components. This is a normal subgroup, so we may apply Mackey's normal subgroup analysis, [M], to construct σ-representations of $\mathcal{G}$. (This subgroup is locally compact and satisfies the conditions needed so the fact that $\mathcal{G}$ itself is not locally compact does not matter.) It is easy to check that the restriction of σ to $\mathcal{C}$ is identically 1, so one starts by taking an ordinary representation, χ of $\mathcal{C}$. All χ on the same $\mathcal{G}$-orbit give equivalent representations, where, since $\mathcal{G}$ is abelian, the action is given in terms of $\tilde{\sigma}(\xi_1, \xi_2) = \sigma(\xi_1, \xi_2)/\sigma(\xi_2, \xi_1)$ by

$$(\xi \cdot \chi)(c) = \tilde{\sigma}(c, \xi)\chi(c).$$

The action of $\mathcal{G}$ on the irreducibles of $\mathcal{C}$ is readily seen to be transitive, so that we may as well take χ to be the trivial representation. Further, the stabiliser of any χ is the connected component of the identity, $\mathcal{G}_0$, to which the trivial representation may be trivially extended.

The restriction of σ to $\mathcal{G}_0$ is constant on $\mathcal{C}$-cosets, and so also defines a cocycle on the quotient $\mathcal{G}_0/\mathcal{C}$. Following the Mackey algorithm we can construct any irreducible σ-representation of $\mathcal{G}$ by choosing an irreducible σ- representation Φ of $\mathcal{G}_0/\mathcal{C}$ lifting it to a σ- representation Φ' of $\mathcal{G}_0$ and then inducing this to $\mathcal{G}$. Thus it remains only to choose Φ, and this is where the forms enter.

Any element $\xi \in \mathcal{G}_0$ can be mapped to the form $d\xi/\xi$ on the boundary $\partial\Sigma_1$, and this induces an injection on $\mathcal{G}_0/\mathcal{C}$. The image consists of the subspace of forms whose integral round every boundary component vanishes. The complexification of this space can be regarded as a subspace Γ_0 of the limit of $\Gamma(U_1 \cap U_2, \mathcal{O}(K))$ as U_1 and U_2 shrink down to Σ_1 and Σ_2 respectively. The cocycle σ on $\mathcal{G}_0/\mathcal{C}$ reduces to

$$\sigma(e^{if}, e^{ig}) = \exp(is(idf, idg)/2),$$

where s is the symplectic form on the space Γ_0 given by

$$s(\xi, \eta) = \frac{1}{2\pi}\int_{\partial\Sigma_1} g\xi,$$

where g is a (multi-valued) function whose differential $dg = \eta$. One readily checks that this is well-defined and indeed symplectic (cf [JKL]). We now apply the Mayer-Vietoris sequence

$$0 \to \Gamma(\Sigma, \mathcal{O}(K)) \to \Gamma(U_1, \mathcal{O}(K)) \oplus \Gamma(U_2, \mathcal{O}(K)) \to \Gamma(U_1 \cap U_2, \mathcal{O}(K)) \to H^1(\Sigma, \mathcal{O}(K)) \to 0.$$

The vanishing condition on boundary circles means that we can restrict attention to elements of $\Gamma(U_1 \cap U_2, \mathcal{O}(K))$ which map to 0 in $H^1(\Sigma, \mathcal{O}(K))$, so that, introducing the suffix 0 to signify the vanishing of the boundary integrals we can reduce the sequence to

$$0 \to \Gamma_0(\Sigma, \mathcal{O}(K)) \to \Gamma_0(U_1, \mathcal{O}(K)) \oplus \Gamma_0(U_2, \mathcal{O}(K)) \to \Gamma_0(U_1 \cap U_2, \mathcal{O}(K) \to 0.$$

($H^1(\Sigma, \mathcal{O}(K))$ is a one-dimensional space spanned by the meromorphic form which is denoted by τ in [JKL].

Before going any further it is useful to introduce some notation. Let us write $A_0 = \partial_0\Sigma_1, \ldots, A_p = \partial_p\Sigma_1$ for the boundary components of Σ_1. Although it would be possible to work more generally we shall work only with the case where Σ is the Schottky double of Σ_1. We may then choose canonical homology generators $A_1, A_2, \ldots, A_g$ and $B_1, \ldots, B_g$, so that the first p A-cycles are boundary circles and the rest fall into pairs $A_{p+2k-1} = \tilde{A}_{p+2k}$, for $k = 1, \ldots, (g-p)/2$. Also for $j = 1, \ldots, p$, B_j is a cycle which intersects A_0 and A_j just once and meets no other A-cycles. Let $\omega_1, \ldots, \omega_g$ be a basis for the cohomology such that

$$\int_{A_k} \omega_j = 2\pi i \delta_{jk} \qquad \int_{B_k} \omega_j = \tau_{jk},$$

where τ_{jk} is the period matrix. These conventions agree in most important aspects with those of [F], Section 6. As there we have $\tilde{\omega}_{p+2k-1} = -\overline{\omega}_{p+2k}$ for $k = 1, \ldots, (g-p)/2$.

The ω_j span the g-dimensional space $\Gamma(\Sigma, \mathcal{O}(K))$, but clearly $\omega_1, \ldots, \omega_p$ are excluded from the subspace $\Gamma_0(\Sigma, \mathcal{O}(K))$ by the vanishing condition on $A_0, \ldots, A_p$. Since for holomorphic forms Cauchy's Theorem means that the integral round the whole boundary $\partial\Sigma_1$ is 0, the vanishing of the integrals round A_j for $j = 1, \ldots, p$ is sufficient to give a vanishing integral round A_0 as well. Thus the remaining forms ω_j for $j = p+1, \ldots, g$ are in Γ_0, and they span $\Gamma_0(\Sigma, \mathcal{O}(K))$.

Part of the problem with using the Mayer-Vietoris sequence to get a Fock decomposition is that the subspaces $\Gamma_0(\Sigma_j, \mathcal{O}(K))$ are not isotropic with respect to the symplectic form. As in [JKL] we have

$$2\pi s(\xi, \eta) = \int_{\partial\Sigma_1} g df = \sum \{ \int_{A_j} g\xi + \int_{B_j} g\xi - \int_{A_j} g\xi - \int_{B_j} g\xi \},$$

which may be rewritten as

$$\sum \{ \int_{A_j} \xi \int_{B_j} \eta - \int_{A_j} \eta \int_{B_j} \xi \}.$$

Inserting $\xi = \omega_k$, $\eta = \omega_l$ one obtains

$$s(\omega_k, \omega_l) = i \sum \{ \delta_{jk} \tau_{jl} - \delta_{jl} \tau_{jk} \}.$$

If k and l are both positive and of the same parity or if neither is positive then s vanishes. If $k-p$ is positive and odd but $l-p$ is not then

$$s(\omega_k, \omega_l) = i\tau_{kl},$$

which with the antisymmetry of s determines the form completely. The negativity of the real part of τ means that this is non-degenerate on $\Gamma_0(\Sigma, \mathcal{O}(K))$. The formula for s suggests that we should concentrate attention on the isotropic subspaces Γ_j of forms holomorphic on Σ_j whose integrals round all A-cycles in Σ_j vanish, $(j = 1, 2)$. Moreover, the images of these two subspaces still span Γ_0. so that Γ_0 can be identified with the direct sum of the isotropic subspaces Γ_1 and Γ_2. Furthermore, these spaces are images of each other under the map induced by the involution ϕ. We may therefore carry out a standard Fock construction to obtain a cyclic σ-representation Φ of $\mathcal{G}_0/\mathcal{C}$ with

$$< \Omega_\Phi, \Phi(\alpha)\Omega_\Phi >= \exp(-\frac{i}{2}s(\tilde{\alpha}, P_1\alpha)),$$

where Ω_Φ is the cyclic vector and P_1 is the projection onto Γ_1. We shall write $\mathcal{H}_\Phi$ for the corresponding representation space, and let

$$\|\alpha\|^2 = s(\tilde{\alpha}, P_1\alpha).$$

Just as the projection for fermions can be written in terms of the Szegö kernel, so the projection here can be written in terms of the Bergmann kernel $\mathcal{B}$ of the surface Σ, [F, Chapter 6]. In fact

$$\|\alpha\|^2 = \int_{\Sigma_1} \mathcal{B}_1(x,y)\tilde{\alpha}(x)\alpha(y)$$

where $\mathcal{B}_1 = p_1\mathcal{B}$ is obtained by projecting out the components of ω_{p+2k-1} for

$$k = 1, \ldots, (g-p)/2$$

.

The conclusion of the preceding discussion is that we may induce an irreducible σ-representation $\Gamma = \mathrm{ind}(\Phi')$ of $\mathcal{G}$ on the space $\mathcal{H} = l^2(\mathcal{G}/\mathcal{G}_0, \mathcal{H}_\Phi)$. (Since $\mathcal{G}_0$ is the connected component of the identity we have $\mathcal{G}/\mathcal{G}_0 \cong \mathbf{Z}^{p+1}$ and $\mathcal{H} \cong l^2(\mathbf{Z}^{p+1}) \otimes \mathcal{H}_\Phi$. Since the cycle A_0 plays a slightly different role from the others it is convenient to separate this out and think of $\mathcal{G}/\mathcal{G}_0$ as $\mathbf{Z} \times \mathbf{Z}^p$, and a typical element as (m_0, m). If we write a typical element,ξ of $\mathcal{G}$ as (n_0, n, ν_0, ν, F), where $(n_0, n) \in \mathbf{Z} \times \mathbf{Z}^p$ gives the winding numbers round the boundary components, (ν_0, ν) gives the constant terms in the Fourier series of $\log\xi$ on the boundary components, and F gives the remainder of the Fourier series on the boundary then the induced representation may be written explicitly in the form

$$(\Gamma(\xi)\psi)(m_0, m) = e^{i((m_0-a_0)\alpha_0 + (m-a)\cdot\alpha)}\Phi(F)\psi(m_0 - a_0, m - a),$$

where ψ is an element of $l^2(\mathbf{Z}^{p+1}) \otimes \mathcal{H}_\Phi$.

For any half-period e consider the vector $\Omega \in \mathcal{H} = l^2(\mathbf{Z}^{p+1}) \otimes \mathcal{H}_\Phi$ given by

$$\Omega(m_0, m) = K\delta(m)\exp(-(m+e)\tilde{\tau}(m+e)/4)\Omega_\Phi,$$

where K is a normalisation factor, $\delta(m)$ is 1 if the sum of m_0 and the other components of m vanishes and otherwise is 0, and $\tilde{\tau}$ is the $p \times p$ part of the period matrix which refers to boundary cycles only. This is essentially the prescription in [E]. (It is not clear there, however, that the weights refer only to the boundary cycles.)

Applying the induced representation to this vector gives

$$(\Gamma(\xi)\Omega)(m_0, m) = K\delta(m-a)e^{i((m_0-a_0)\alpha_0 + (m-a)\cdot\alpha)}e^{-(m-a+e)\tilde{\tau}(m-a+e)/4}\Phi(F)\Omega_\Phi.$$

so that

$$< \Omega, \Gamma(\xi)\Omega >= |K|^2 \delta(a) e^{-a\tilde{\tau}a/8} e^{-\|F\|^2/4} \sum e^{i((m-a)\cdot(\alpha-\alpha_0)} e^{-(m-\frac{1}{2}a+e)\tilde{\tau}(m-\frac{1}{2}a+e)/2},$$

where the sum runs over all $m \in \mathbf{Z}^p$ and $\alpha - \alpha_0$ is to be interpreted as the vector each of whose components is α_0 less than the corresponding component of α. This expression may be rewritten in terms of theta-functions as

$$|K|^2 \delta(a) e^{-a\tilde{\tau}a/8} e^{-\|F\|^2/4} \theta_{\tilde{\tau}} \begin{bmatrix} e - \frac{1}{2}a \\ 0 \end{bmatrix} (\alpha - \alpha_0).$$

Taking ξ to be the identity gives

$$|K|^2 = \theta_{\tilde{\tau}} \begin{bmatrix} e \\ 0 \end{bmatrix} (0)^{-1},$$

so that

$$< \Omega, \Gamma(\xi)\Omega >= \delta(a) e^{-a\tilde{\tau}a/8} e^{-\|F\|^2/4} \theta_{\tilde{\tau}} \begin{bmatrix} e - \frac{1}{2}a \\ 0 \end{bmatrix} (\alpha - \alpha_0) / \theta_{\tilde{\tau}} \begin{bmatrix} e \\ 0 \end{bmatrix} (0)^{-1}.$$

The choice of state Ω may seem a little arbitrary but there is an alternative more geometric way of constructing it. Formally, Ω is an eigenvector of any element ξ in the complexified group such that $\alpha - \frac{1}{2}i\tilde{\tau}a$ vanishes. We defer the details to our promised fuller account. Note that we could extend many of these results to the group of maps from $\partial\Sigma_1$ to a compact Lie group G. We describe one such extension in the next section.

6. Geometric construction of the cocycle

In keeping with our geometric approach it is worth observing here that the cocycle which was introduced by an analytic argument in the earlier sections also has a geometric construction. So we begin with a Riemann surface Σ and compact Lie group G and introduce

$$\Sigma G = C^\infty(\Sigma, G).$$

Assume that $\partial\Sigma$ has just one component. The generalisation to many components is straight forward. If LG denotes the loop group then taking boundary values defines a map

$$b \colon \Sigma G \to LG.$$

We want to show how to define a lift of this map to a map $\hat{b}$ into the central extension of the loop group.

Fix a generator ω of $H^3(G, 2\pi i\mathbf{Z})$ and let F_ω be the two form on LG obtained by pulling back under the evaluation map $ev \colon S^1 \times LG \to G$ and then integrating over the circle. That is

$$F_\omega = \int_{S^1} ev * (\omega).$$

Denote by F the usual right invariant two-form and let ∇ and ∇_ω be connections whose curvatures are F and F_ω repectively.

Let $f\colon \Sigma \to G$. Define a cap for f to be a map γ from the disk D into G which agrees with f on the boundary of Σ and is the identity at the center of the disk. Note that γ can also be regarded as a path joining the identity to the point $b(f)$. Denote by $\Sigma \cup D$ the closed Riemann surface obtained by identifying the boundaries of Σ and D. Then the pair f and γ defines a map from $\Sigma \cup D$ into G which we shall denote by $f * \gamma$. Define $\alpha(f, \gamma) \in U(1)$ to be the Wess-Zumino term of $f * \gamma$. That is we extend $f * \gamma$ to the interior of $\Sigma \cup D$, pull back ω and integrate and then take the exponential. This is independent of the choice of the extension to the interior. Notice that if we change the capping to γ' then the two caps together form a loop in LG. Call this $\gamma * \gamma'$. Then we have

$$\alpha(f, \gamma) = \alpha(f, \gamma') hol(\gamma * \gamma', \nabla_\omega).$$

where *hol* denotes the holonomy along $\gamma * \gamma'$.

The definition of the lift of b is as follows. Take f and a cap γ. Then as we have seen γ is a path from e to $b(f)$. Lift this to a horizontal path $\hat{\gamma}$ for ∇_ω. Then define

$$\hat{b}(f) = \hat{\gamma}(1)\alpha(f, \gamma).$$

If we change the cap then we have seen that the α changes by the holonomy of ∇_ω along the loop formed by the two caps. But that is also what the difference between $\hat{\gamma}(1)$ and $\hat{\gamma}'(1)$ is so the two changes cancel.

The map b pulls back the central extension of the loop group to a central extension of ΣG. The point of $\hat{b}$ is that it defines a splitting of this central extension and hence a co-cycle which we shall now calculate. Call the pull back central extension $\hat{\Sigma G}$. Then it is the subgroup of $\Sigma G \times \hat{LG}$ of pairs (f, x) where $b(f) = \pi(x)$ where $\pi\colon \hat{LG} \to LG$ is the projection. The splitting is the map

$$\Sigma G \to \hat{\Sigma G}$$

$$f \mapsto (f, \hat{b}(f)).$$

In other words we identify the product $U(1) \times \Sigma G$ with $\hat{\Sigma G}$ by $(f, z) \mapsto (\hat{b}(f) z)$. It is straightforward to calculate that the semi-direct product this induces is

$$(f, 1)(g, 1) \mapsto (fg, c(f, g))$$

where

$$c(f, g) = \frac{\hat{b}(f)\hat{b}(g)}{\hat{b}(fg)}.$$

We would like to calculate this in terms of things we know about on Σ and G. To do this choose caps in γ_f and γ_g for f and g and note that $\gamma_f\gamma_g$ is a cap for fg. Then from the definition of $\hat{b}$ we have

$$c(f, g) = \frac{\hat{\gamma}_f(1)\hat{\gamma}_g(1)\alpha(f, \gamma_f)\alpha(g, \gamma_g)}{\widehat{\gamma_f\gamma_g}(1)\alpha(fg, \gamma_f\gamma_g)}.$$

We can write this expression in terms of the co-cycle used in [Mu]. Note first though that in that construction the connection ∇ was used. The two connections ∇ and ∇_ω differ by a one-form η which can be explicitly calculated (see [PS]). If we denote by c_M the cocycle used in [Mu] we obtain

$$c_(f,g) = c_M(\gamma_f,\gamma_g)\frac{\alpha(f,\gamma_f)\alpha(g,\gamma_g)}{\alpha(fg,\gamma_f\gamma_g)}\exp(\int_{\gamma_f}\eta + \int_{\gamma_g}\eta - \int_{\gamma_f\gamma_g}\eta).$$

The viewpoint of this section is clearly influenced by the Wess-Zumino-Witten model and indeed our results have some bearing on that field theory however we have not obtained a definitive picture as yet.

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Acknowledgement We thank Graeme Segal and Michael Singer for several illuminating discussions.