1. Let $S \subset \mathbb{R}$ be a subset of the real numbers.
   (a) Define what it means for $S$ to be bounded above by a number $K$.
   (b) If $S$ is bounded above define the least upper bound $\sup(S)$.
   (c) For each of the following sets say whether they are bounded above or below or not bounded above or below. Give a bound if there is one. You are not required to prove anything.
     (i) $\left\{ \frac{1}{n^2} \mid n = 1, 2, 3, \ldots \right\}$
     (ii) $\left\{ x \mid x^2 - 9 < 0 \right\}$
     (iii) $\left\{ (x + 2002)^6 \mid x \in \mathbb{R} \right\}$
   (d) Find the sup or inf as indicated below. You are not required to prove anything.
     (i) $\inf\left\{ x \mid 0 < x < 2002 \right\}$
     (ii) $\inf\left\{ \frac{1}{n+2002} \mid n = 1, 2, 3, \ldots \right\}$
     (iii) $\sup\{\cos(x) \mid x \in \mathbb{R}\}$

2. (a) Define what it means for a sequence of real numbers $\left\{ x_n \right\}_{n=1}^{\infty}$ to have a limit $x \in \mathbb{R}$.
   (b) Assume that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} x_n = y$. Show that $x = y$.
   (c) Assume that $\lim_{n \to \infty} x_n = x$ and $\lambda \in \mathbb{R}$. If $z_n = \lambda x_n$ for all $n$ show that $\lim_{n \to \infty} z_n = \lambda x$.

3. Let $V$ be a real vector space.
   (a) Define what it means for a map $\| \| : V \to \mathbb{R}$ to be a norm on $V$.
   (b) Define $\| \| : \mathbb{R}^n \to \mathbb{R}$ by $\| x \|_1 = |x^1| + |x^2| + \cdots + |x^n|$ for $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$. Show that $\| \|_1$ is a norm.
   (c) Consider the function $\| \| : \mathbb{R}^2 \to \mathbb{R}$ defined by $\| x \| = |x^1|$ where $x = (x^1, x^2)$. Is this a norm? Explain why or why not.

4. Let $(X, d)$ be a metric space.
   (a) What is the definition of an open ball in $X$?
(b) What is the definition of an open set in $X$?
(c) Use your definitions in (a) and (b) to that if $U_1$ and $U_2$ are open sets so also is $U_1 \cap U_2$.

5. Let $T : X \to X$ be a function where $(X, d)$ is a metric space.
(a) State what it means for $T$ to have a fixed point.
(b) State what it means for $T$ to be a contraction.
(c) State the Contraction Mapping Theorem concerning fixed points of contractions. (You do not need to define any terms.)
(d) Show that if $|\lambda| < 1$ then there is a unique continuous function $f$ on $[0, 1]$ such that
\[ f(x) = \sin(x) + \lambda \int_0^x f(y) \, dy. \]
You may assume that if $g$ is a continuous function on $[0, 1]$ then $h(x) = \int_0^x g(y) \, dy$ is continuous and that $C[0, 1]$ is complete with the uniform or supremum norm $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 1]\}$.

6. Let $(X, d)$ be a metric space.
(a) Define an open cover of a subset $A \subseteq X$.
(b) Define what it means for a subset $A \subseteq X$ to be compact.
(c) Show that a subset of a metric space with the discrete metric
\[ d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} \]
which is compact is finite.

7. (a) State the Heine-Borel theorem that characterises the sequentially compact subsets of $\mathbb{R}^n$.
(b) Which of the following subsets are sequentially compact and which are not? Give reasons.
   (i) $B(x, 2002) \subseteq \mathbb{R}^n$
   (ii) $[0, 2002] \subseteq \mathbb{R}$
   (iii) $\{n^3 \mid n = 1, 2, \ldots\} \subseteq \mathbb{R}$
   (iv) $\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 2002\} \subseteq \mathbb{R}^3$

8. Let $(X, d)$ be a metric space.
(a) Show that if $x, y, z \in X$ then
\[ |d(x, y) - d(x, z)| \leq d(y, z). \]
(b) Define what it means for a function $f : X \to \mathbb{R}$ to be continuous at a point $x \in X$.
(c) Fix a point $x_0 \in X$. Show that the function $f : X \to \mathbb{R}$, defined by $f(x) = d(x_0, x)$, is continuous at every point $x \in X$.

9. Let $(V, \| \|)$ be a normed vector space.
(a) If $T : V \to \mathbb{R}$ is linear, show that $T$ is continuous if there is a constant $C > 0$ such that $|T(v)| \leq C\|v\|$ for all $v \in V$.
(b) If $T : V \to \mathbb{R}$ is linear and continuous how do we define $\|T\|$ the norm of $T$?
(c) Let $C[0, 1]$ be the space of all continuous functions on $[0, 1]$ with the uniform norm $\|f\|_\infty = \sup\{|f(x)| \mid x \in [0, 1]\}$. Find the norm of $T : C[0, 1] \to \mathbb{R}$ where $T(f) = \int_0^1 f(t) \, dt$.

10. Let $(H, \langle \, , \rangle)$ be a real Hilbert space.
(a) State the Cauchy-Schwarz inequality.
(b) If $v \in H$ define $L_v : H \to \mathbb{R}$ by $L_v(w) = \langle v, w \rangle$. Show that $L_v$ is linear, continuous and find its norm.
(c) Show that $\ker(L_v) = \{w \in H \mid L_v(w) = 0\}$ is a closed subspace of $H$.
(d) Show that $\ker(L_v)^\perp = \{\lambda v \mid \lambda \in \mathbb{R}\}$.

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