

## Summary of Vector Spaces and Linear Functions

If you have done Algebra II this should nearly all be revision. If you have only done Mathematics I the main difference is that here we are interested in abstract vector spaces whereas in Mathematics I every vector space you looked at was a subspace of  $\mathbb{R}^n$  so we never admitted to the possibility of infinite-dimensional vector spaces.

**Definition 1.1.** Let  $K$  be  $\mathbb{R}$  or  $\mathbb{C}$ . A *real (or complex) vector space*  $V$  is a set with two operations as follows. We can add two vectors together (vector addition):

$$\begin{aligned} V \times V &\rightarrow V \\ (v, w) &\mapsto v + w \end{aligned}$$

and we can multiply a vector by a real or complex number (scalar multiplication)

$$\begin{aligned} K \times V &\rightarrow V \\ (\lambda, v) &\mapsto \lambda v \end{aligned}$$

These satisfy a number of axioms:

- (1)  $u + v = v + u$ ,
- (2)  $(u + v) + w = u + (v + w)$ ,
- (3) there is an element  $0 \in V$  such that  $0 + v = v = v + 0$ ,
- (4) for each  $v \in V$  there is a  $-v \in V$  such that  $v + (-v) = 0$ ,
- (5)  $(r + s)v = rv + sv$ ,
- (6)  $r(sv) = (rs)v$ ,
- (7)  $r(u + v) = ru + rv$ ,
- (8)  $1v = v$ ,
- (9)  $0v = 0$ , and
- (10)  $(-1)v = -v$  for every  $u, v, w \in V$  and  $r, s \in K$ .

*Example 1.1.* The examples you have seen in Mathematics I are  $\mathbb{R}^n$  and  $\mathbb{C}^n$  and subspaces of these.

*Example 1.2.*  $\mathcal{P}$  the set of all real or complex valued polynomials and  $\mathcal{P}_n$  the set of all real or complex valued polynomials of degree less than or equal to  $n$ .

*Example 1.3.* If  $X$  is any set then the collection of all functions from  $X$  to  $\mathbb{R}$  or  $\mathbb{C}$  is a real or complex vector space. The addition is  $(f+g)(x) = f(x)+g(x)$  and scalar multiplication is  $(rf)(x) = rf(x)$ . Notice that we can regard  $\mathbb{R}^n$  as all functions from  $\{1, 2, \dots, n\}$  into  $\mathbb{R}$  and similarly  $\mathbb{C}^n$ .

**Definition 1.2.** A subset  $W$  of a vector space  $V$  is called a *subspace* if it contains  $0$  and if for any  $v, w \in W$  we have  $v + w \in W$  and if for any  $r \in K$  and  $v \in W$  we have  $rv \in W$ .

**Definition 1.3.** If  $S$  is a subset of a vector space  $V$  we define the span of  $S$ , denoted  $\text{Span}(S)$  by

$$\text{Span}(S) = \{r_1v_1 + \dots + r_kv_k \mid r_1, \dots, r_k \in K, v_1, \dots, v_k \in S\}.$$

**Proposition 1.1.**  $\text{Span}(S)$  is a subspace of  $V$  and  $S \subset \text{Span}(S)$ . If  $W$  is a subspace of  $V$  and  $S \subset W$  then  $\text{Span}(S) \subset W$ .

**Definition 1.4.** A subset  $S$  of a vector space  $V$  is called *linearly dependent* if there are vectors  $v_1, \dots, v_k$  in  $S$  and scalars  $r_1, \dots, r_k$  in  $K$ , which are not all zero, such that

$$r_1v_1 + r_2v_2 + \dots + r_kv_k = 0.$$

$S$  is called *linearly independent* if it is not linearly dependent.

**Definition 1.5.** A subset  $S$  of a vector space  $V$  is called a *basis* if it is linearly independent and spans  $V$  that is  $\text{Span}(S) = V$ .

*Example 1.4.* Let  $e_i$  be the vector in  $\mathbb{R}^n$  or  $\mathbb{C}^n$  which has all entries zero except for the  $i$ th entry which is 1. Then  $e_1, \dots, e_n$  is a basis for  $\mathbb{R}^n$  or  $\mathbb{C}^n$  called the standard basis.

*Example 1.5.* The set  $\{1, x, x^2, \dots, x^n\}$  is a basis for  $\mathcal{P}_n$  and  $\{1, x, x^2, \dots\}$  is a basis for  $\mathcal{P}$ .

**Proposition 1.2.** *If  $V$  has a basis which is a finite set then the number of elements in the basis is the same for every basis of  $V$ .*

**Definition 1.6.** If  $V$  has a finite dimensional basis then we call the number of elements in it the *dimension* of  $V$  and denote it  $\dim(V)$ .

If  $V$  does not have finite dimensional basis then we call it infinite dimensional.

*Example 1.6.*  $\mathbb{R}^n$  and  $\mathbb{C}^n$  have dimension  $n$ .  $\mathcal{P}_n$  has dimension  $n + 1$ .  $\mathcal{P}$  is infinite dimensional.

**Definition 1.7.** Let  $V$  and  $W$  be vector spaces. A map  $T: V \rightarrow W$  is called a linear function (or linear transformation) if  $T(rv) = rT(v)$  and  $T(v + w) = T(v) + T(w)$  for every  $r \in K$  and  $v, w \in V$ .

**Definition 1.8.** Let  $T: V \rightarrow W$  be a linear transformation. We define the kernel,  $\ker(T)$  and image,  $\text{im}(T)$  by

$$\begin{aligned}\ker(T) &= \{v \in V \mid T(v) = 0\} \\ \text{im}(T) &= \{T(v) \mid v \in V\}\end{aligned}$$

**Proposition 1.3.** *Let  $T: V \rightarrow W$  be a linear transformation. Then  $\ker(T)$  and  $\text{im}(T)$  are subspaces. If  $V$  and  $W$  are finite dimensional then*

$$\dim \ker(T) + \dim \text{im}(T) = \dim V.$$

**Corollary 1.4.** *Let  $T: V \rightarrow W$  be a linear transformation between finite dimensional vector spaces. Then the following are equivalent*

- (1)  $T$  has an inverse
- (2)  $T$  is one to one, and
- (3)  $T$  is onto.

Let  $T: V \rightarrow W$  be a linear transformation. Assume that  $v_1, \dots, v_n$  is an ordered basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Define a matrix  $T_{ij}$  by

$$T(v_i) = \sum_{j=1}^m T_{ij} w_j.$$

We call  $T_{ij}$  the matrix representation of  $T$  with respect to the ordered bases  $v_1, \dots, v_n$  and  $w_1, \dots, w_m$ . Notice that  $\dim \ker(T)$  is the nullity of  $T_{ij}$  or the dimension of the nullspace of  $T_{ij}$  and  $\dim \text{im}(T)$  is the column rank or rank of  $T_{ij}$ . The result in Proposition 1.3 is taught in Mathematics I as the Rank Theorem and says that  $\text{rank}(A) + \text{nullity}(A) = n$  for any  $n \times m$  matrix  $A$ .

**Proposition 1.5.** *Let  $T: V \rightarrow W$  be a linear transformation. If  $U \subset V$  is a subspace then  $T(U) \subset W$  is a subspace. If  $U \subset W$  is a subspace then  $T^{-1}(U) \subset V$  is a subspace.*

*Example 1.7.* Note that  $\ker(T) = T^{-1}\{0\}$  and  $\text{im}(T) = T(V)$ .