1. Let $d_1$ and $d_2$ be metrics on a space $X$. Show that:
   (a) $d(x, y) = d_1(x, y) + d_2(x, y)$ and,
   (b) $d(x, y) = \max\{d_1(x, y), d_2(x, y)\}$ are metrics on $X$.

2. Let $(X, d)$ be a metric space. Show that for any points $x \neq y$ in $X$ there is a $\delta > 0$ such that $x \notin B(y, \delta)$. Using this, or otherwise, show that any subset of $X$ consisting of just a single point is closed. Show that a subset of $X$ consisting of a finite number of points is closed.

3. Let $(V, \langle \ , \rangle)$ be an inner product space. Show that if $v_n \rightarrow v$ and $w \in V$ then $\lim_{n \to \infty} \langle v_n, w \rangle = \langle v, w \rangle$. [Hint: Cauchy's inequality.]

4. Let $(X, d)$ be a metric space and $A \subset X$ a subset. Show that $x \in \overline{A}$ if and only if there is a sequence $\{a_n\}_{n=1}^{\infty}$ with each $a_n \in A$ and $a_n \rightarrow x$. [You should be able to get this from what we have proved in Lectures.]

5. Let $W$ be a subspace of a normed vector space $(V, \|\ |)$. Show that the closure of $W$ is a subspace. [Hint: Consider the previous question.]

6. Consider the set $X = \{0, 1\}$ with the discrete metric. What is the closed ball around 0 of radius 1, i.e. $B(0, 1)$? What is the closure of the open ball around 0 of radius 1, i.e. $B(0, 1)$? Are they the same?

7. Consider $C[0,1]$ with the uniform norm. Let
   $$Y = \{f \in C[0,1] \mid f(1/3) = f(2/3) = 0\}.$$
   Show that $Y$ is closed.

8. Consider a set $X$ with the discrete metric. Show that a sequence in $X$ is Cauchy if and only if there is an $N$ such that for all $n, m \geq N$ $x_n = x_m$. Similarly show that $x_n \rightarrow x$ in the discrete metric if and only if there is an $N$ such that for all $n \geq N$ $x_n = x$. Deduce that a set with the discrete metric is a complete metric space.
1. Let $(X_1, d_1)$ and $(X_2, d_2)$ be metric spaces. Consider $X = X_1 \times X_2$ and define $d: X \times X \to \mathbb{R}$ by

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

Show that $d$ is a metric on $X$.

2. Let $(X, d)$ be a metric space. Show that if $x$ and $y$ are distinct points in $X$ then there is a $\delta > 0$ such that the balls $B(x, \delta)$ and $B(y, \delta)$ do not intersect.

3. Let $(V, \| \cdot \|)$ be a normed vector space. If $\{v_n\}_{n=1}^{\infty}$ is a convergent sequence of vectors with $v_n \to v$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a convergent sequence of numbers with $\lambda_n \to \lambda$ show that the sequence $\{\lambda_n v_n\}_{n=1}^{\infty}$ converges to $\lambda v$. [Hint: Think about the proof when $\{v_n\}_{n=1}^{\infty}$ is also a sequence of real numbers.]

4. Let $\ell_\infty$ be the set of all bounded sequences with the uniform norm: $\|x\| = \sup \{|x_1|, |x_2|, |x_3|, \ldots\}$ (called $b$ in Assignment 2). Let $Y$ be the subset of $\ell_\infty$ of all sequences $x$ such that $x_{100} = 0$. Show that $Y$ is closed.

5. Let $\{x^{(n)}_i\}_{n=1}^{\infty}$ be a sequence in $\ell_\infty$. Note that this means that each $x^{(n)}$ is itself a sequence $x^{(n)} = \{x^{(n)}_i\}_{i=1}^{\infty}$. Show that if $\lim_{n \to \infty} x^{(n)} = x$ where $x = \{x_i\}_{i=1}^{\infty}$ then for every $i$ we have $\lim_{n \to \infty} x^{(n)}_i = x_i$.

Show that the converse is not true. That is find a sequence $\{x^{(n)}_i\}_{n=1}^{\infty}$ such that $\lim_{n \to \infty} x^{(n)}_i = x_i$ for every $i$ but such that $\lim_{n \to \infty} x^{(n)} \neq x$.

6. Show that $\ell_\infty$ is complete.

7. Let $c_x$ be the set of all sequences in $\ell_\infty$ which converge to $x \in \mathbb{R}$. Show that $c_0$ is a closed subspace of $\ell_\infty$. If $x \in \mathbb{R}$ let $\hat{x}$ be the constant sequence $x, x, x, \ldots$. Show that $c_x = c_0 + \hat{x}$ where

$$c_0 + \hat{x} = \{y \in \ell_\infty \mid y = z + \hat{x}, z \in c_0\}.$$

8. Let $f$ be the subspace of all sequences $x$ in $\ell_\infty$ such that there is an $N$ with $x_n = 0$ for all $n \geq N$. Notice that we allow $N$ to vary for different sequences $x$. Show that the closure of $f$ in $\ell_\infty$ is the subspace $c_0$. 