PURE MTH 3002 Topology & Analysis III (3246)

Assignment 1. Due on Monday 24th 2003

In all these questions when you are asked to give a proof you should be careful that you give a logical, carefully laid out argument. It should also be grammatically correct when read out in words. If you have problems or questions please come and see me or email me at michael.murray@adelaide.edu.au.

1. (a) Let \( f: \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = \sin(x) \). If \( A = (0, \pi/2) \) and \( B = [0, 25] \) find \( f(A) \) and \( f^{-1}(B) \).
(b) Let \( f: \mathbb{Z} \to \mathbb{Z} \) be defined by \( f(n) = n + 1 \). Is \( f \) onto or 1-1?
(c) Let \( f: \mathbb{R} \to \mathbb{Z} \) be defined by \( f(x) = \lfloor x \rfloor \) where \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). Is \( f \) onto or 1-1?
(d) Let \( U_n = (-1/n, 1/n) \subset \mathbb{R} \) for \( n \in \mathbb{N} \). Calculate \( \bigcup_{n \in \mathbb{N}} U_n \) and \( \bigcap_{n \in \mathbb{N}} U_n \).

2. For each of the following sets say whether they are bounded above or below or not bounded above or below. You are not required to prove what you say.
(a) \( \{1/n^2 \mid n = 1, 2, 3, \ldots \} \)
(b) \( \{x \mid x^3 - 27 < 0\} \)
(c) \( \{x \mid x^3 - 27 > 0\} \)
(d) \( \{x^2 \mid x \in \mathbb{R}\} \)

3. Find the sup or inf as indicated below. You are not required to prove anything.
(a) \( \inf(−1, 27) \)
(b) \( \sup\{x \mid x^2 - 9 < 0\} \)
(c) \( \inf\{1/n \mid n = 1, 2, 3, \ldots\} \)
(d) \( \sup\{\cos(x) \mid x \in \mathbb{R}\} \)

4. Let \( S \subset \mathbb{R} \) be bounded above. Define \( T = \{2s \mid s \in S\} \subset \mathbb{R} \) show that \( T \) is bounded above and that \( \sup(T) = 2 \sup(S) \).

5. Find the limit of the sequences:
(a) \( x_n = \frac{1}{n^4} \)
(b) \( x_n = \frac{2n + 1}{n^3 - 6n + 1} \)
(c) \( x_n = \frac{1}{n^2} \cos(n) \)

You should use the limit laws proved in class.

6. If \( \{x_n\}_{n=1}^\infty \) is a sequence with limit \( x \) and \( a \in \mathbb{R} \) show that \( \{ax_n\}_{n=1}^\infty \) has limit \( ax \) without using the limit laws from class. I.e construct an argument using \( \epsilon, N \) etc.

7. Let \( a_0, a_1, a_2, \ldots \) be a sequence of zeros and ones, i.e each \( a_i \in \{0, 1\} \). Define a sequence of real numbers \( x_1 = a_0, x_2 = a_0 + a_1/2, x_3 = a_0 + a_1/2 + a_3/2^2 \) etc. Show that \( x_1, x_2, x_3, \ldots \) is a Cauchy sequence.
Tutorial Exercises. Please try them before the tutorial on Wednesday 19th 2003

1. (a) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = x^2 \). If \( A = (0, \pi/2) \) and \( B = [-25, 25] \) find \( f(A) \) and \( f^{-1}(B) \).
(b) Let \( f : \mathbb{Z} \to \mathbb{Z} \) be defined by \( f(n) = n^2 + 10 \). Is \( f \) onto or 1-1?
(c) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by \( f(x) = [x] \) where \([x]\) is the smallest integer greater than or equal to \( x \). Is \( f \) onto or 1-1?
(d) Let \( U_n = (-1/n, n) \subset \mathbb{R} \) for \( n \in \mathbb{N} \). Calculate \( \bigcup_{n \in \mathbb{N}} U_n \) and \( \bigcap_{n \in \mathbb{N}} U_n \).

2. (a) Show that if \( x, y \in \mathbb{R} \) and \( y - x > 1 \) then there is an integer \( n \in \mathbb{Z} \) with \( x < n < y \). [Hint consider the set \( \{ n \in \mathbb{Z} | n < y \} \) and apply the least upper bound axiom.]
(b) Using (a) or otherwise show that if \( x, y \in \mathbb{R} \) with \( x < y \) then there are integers \( p \) and \( q \) with \( q \neq 0 \) and \( x < p/q < y \). That is, between any two real numbers there is a rational number.

3. For each of the following sets say whether they are bounded above or below or not bounded above or below. You are not required to prove what you say.
(a) \( \{ 1/p^2 | p \in \mathbb{Z} \ldots \} \)
(b) \( \{ \sin(x) | x \in \mathbb{R} \} \)
(c) \( \{ x | x^2 - 9 > 0 \} \)
(d) \( \{ x^2 | x \in \mathbb{R} \} \)

4. Find the sup or inf as indicated below. You are not required to prove anything.
(a) \( \sup(5, 56) \cup \{ 100 \} \)
(b) \( \sup\{ x | x^2 - 25 < 0 \} \)
(c) \( \inf\{ 1/(n + 1) | n = 1, 2, 3, \ldots \} \)
(d) \( \sup\{ \sin(x) + 5 | x \in \mathbb{R} \} \)

5. Let \( S \) and \( T \) be subsets of the real numbers which are bounded above. Assume that for every \( s \in S \) there is a \( t \in T \) with \( s < t \). Show that \( \sup(S) \leq \sup(T) \). Give an example where the hypothesis is satisfied but \( \sup(S) = \sup(T) \).

6. (a) Show that for any \( K \in \mathbb{R} \) there is an \( m \in \mathbb{N} \) with \( 2^m > K \). [Hint: \( 2^m = (1 + 1)^m \).]
(b) Find \( \lim_{n \to \infty} 1/2^n \).

7. Let \( a_0, a_1, a_2, \ldots \) be a decimal expansion, that is \( a_0 \) is an integer and \( a_1, a_2, \ldots \) are numbers from the set \( \{0, 1, \ldots, 8, 9\} \). Define a sequence by \( x_1 = a_0 + a_1/10, x_2 = x_1 = a_0 + a_1/10 + a_2/100 \) so that \( x_r = a_0 + \sum_{i=1}^r a_r/10^r \). Show that \( \{x_i\}_{i=1}^{\infty} \) is a Cauchy sequence.

8. If \( \{x_i\}_{i=1}^{\infty} \) and \( \{y_i\}_{i=1}^{\infty} \) are Cauchy sequences show that \( \{x_i + y_i\}_{i=1}^{\infty} \) is a Cauchy sequence.

9. Define a half-cut of the rationals \( \mathbb{Q} \) to be a subset \( S \subset \mathbb{Q} \) satisfying (i) \( S \neq \emptyset \), (ii) \( S \neq \mathbb{Q} \), (iii) if \( x \in S \) and \( y > x \) then \( y \notin S \), and (iv) \( S \) does not have a greatest lower bound.
(a) If \( x \in \mathbb{R} \) show that \( S = (x, \infty) \cup \mathbb{Q} \) is a half-cut,
(b) If \( A, B \subset \mathbb{Q} \) define \( A + B = \{a + b | a \in A, b \in B \} \). Show that if \( S \) and \( T \) are half-cuts so also is \( S + T \).
(c) If \( A, B \subset \mathbb{Q} \) define \( AB = \{ab | a \in A, b \in B \} \). Show that if \( S \) and \( T \) are half-cuts so also is \( ST \).
(d) How would you define \(-S\) and \( 1/S \)?

I am not sure what these sets are officially called. I have called them half-cuts as \((S^c, S)\) is what Dedekind called a cut. It can be shown that the set of all half-cuts satisfies the axioms for the real numbers. The details are in the book Real Analysis by Michael Spivak. For more about Dedekind see http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Dedekind.html.