1 Sets, Functions and Relations

Review basic definitions of a set, inclusion, subset, power set, union, intersection, disjoint and complement. Review basic set laws relating these operations. Indexing sets. Products.

Review basic properties of functions. Definition of one to one or injective and onto or surjective. Composition of functions. Identity function. Inverse of a function.

A function is invertible if and only if it is one to one and onto (bijective). Image and pre-image. Identities relating image and pre-image.

Definition 1.1. A relation \( R \subseteq X \times X \) is called an equivalence relation if

(i) \( xRx \) (reflexive)

(ii) \( xRy \) implies \( yRx \) (symmetric)

(iii) \( xRy \) and \( yRz \) implies \( xRz \) (transitive)

for all \( x, y, z \in X \).

Definition 1.2. If \( R \) is an equivalence relation on \( X \) then we define the equivalence class containing \( x \) by

\[
[x] = \{y \in X | xRy\}.
\]

Definition 1.3. Let \( X \) be a set. A partition of \( X \) is a collection \( \{U_\alpha | \alpha \in I\} \) of subsets of \( X \) satisfying

(i) \( X = \bigcup_{\alpha \in I} U_\alpha \)

(ii) \( U_\alpha \cap U_\beta = \emptyset \) whenever \( \alpha \neq \beta \).

Proposition 1.1. The equivalence classes of an equivalence relation partition a set.

Definition 1.4. Two sets \( X \) and \( Y \) are said to have the same cardinality if there is a one to one and onto function \( f: X \to Y \).

Definition 1.5. A set \( X \) is called countable if it has the same cardinality as the natural numbers \( \mathbb{N} \).

Lemma 1.2. An infinite subset of a countable set is countable.

Proposition 1.3. The set \( \mathbb{Z} \times \mathbb{N} \) is countable.

Corollary 1.4. The rational numbers are countable.

Proposition 1.5. The set \( (0,1) \subseteq \mathbb{R} \) is not countable.

Corollary 1.6. Any \( (a,b) \subseteq \mathbb{R} \) is uncountable and hence any interval in \( \mathbb{R} \) is uncountable. In particular \( \mathbb{R} \) is uncountable.

2 The real numbers

2.1 Structure of the real numbers

Definition 2.1. The natural numbers are the set \( \mathbb{N} = \{1, 2, \ldots\} \).

Definition 2.2. The integers are the set \( \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \).
Define a relation on the set $\mathbb{Z} \times \mathbb{Z} - \{0\}$ by $(a, b) \sim (c, d)$ if $ad = bc$. This is an equivalence relation. We denote the equivalence class of $(a, b)$ by

$$\frac{a}{b}$$

and call the set of all equivalence classes the **rational numbers** denoted $\mathbb{Q}$. We define an addition and multiplication of rational numbers by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$ 

If $r \in \mathbb{Z}$ we denote $\frac{r}{1}$ by just $r$.

The real numbers $\mathbb{R}$ can be defined from the rational numbers and we may do this in an exercise or you can find it in the references. In class we will content ourselves with presenting the properties of the real numbers.

Firstly the real numbers have two algebraic operations $+$ and $\times$ (called addition and multiplication) and are a **field** with respect to these operations. First let us be precise what we mean by operations. These are two maps

$$+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad (x, y) \mapsto x + y$$

and

$$\times: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \quad (x, y) \mapsto xy.$$  

The axioms of a field are:

**Axioms for Addition:** for all $a, b$ and $c$ in $\mathbb{R}$.

2. Associativity: $a + (b + c) = (a + b) + c$.
3. Identity: There is a number $0$ such that $a + 0 = a$.
4. Inverses: There is a number $-a \in \mathbb{R}$ such that $a + (-a) = 0$.

**Axioms for Multiplication:** for all $a, b$ and $c$ in $\mathbb{R}$

5. Commutativity: $a \cdot b = b \cdot a$.
6. Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
7. Identity: There is a number called $1$, which does not equal $0$, such that $a \cdot 1 = a$.
8. Inverses: If $a \neq 0$, there is a number $a^{-1} \in \mathbb{R}$ such that $a \cdot (a^{-1}) = 1$.

**Distributive Axiom**

9. If $a, b$ and $c$ are in $\mathbb{R}$, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

As well as these algebraic operations the real numbers have a total ordering denoted $<$. This is usually defined as follows.

**Order axiom.** We have a subset $\mathbb{R}_+ \subseteq \mathbb{R}$ called the positive real numbers which satisfies:

(i) for all $a \in \mathbb{R}$ exactly one of the alternatives $a = 0$, $a \in \mathbb{R}_+$, or $-a \in \mathbb{R}_+$ holds, and

(ii) for all $a, b \in \mathbb{R}_+$, $a + b \in \mathbb{R}_+$ and $ab \in \mathbb{R}_+$.

The ordering on the real numbers is then defined by saying that $a > b$ if $a - b \in \mathbb{R}_+$. This is a total ordering.

The final axiom of the real numbers is called the least upper bound axiom. To define this we need a few extra definitions. Let $S \subseteq \mathbb{R}$. We say that $S$ is bounded above if there is a $K \in \mathbb{R}$ such that for all $s \in S$ we have $s \leq K$. We call such a $K$ an upper bound for $S$. We say that $K_0 \in \mathbb{R}$ is a least upper bound for $S$ if $K_0$ is an upper bound for $S$ and for any other upper bound $K$ for $S$ we have $K_0 \leq K$. The least upper bound axiom says:

**Least upper bound axiom.**

Any non-empty subset $S$ of the real numbers which is bounded above has a least upper bound.

Finally we note that $\mathbb{R}$ contains $\mathbb{Q}$ and that the $+, \times$ and $<$ for $\mathbb{R}$ are the same as the usual ones on $\mathbb{Q}$ when applied to elements of $\mathbb{Q}$.

**Lemma 2.1 (Archimedean Property).** If $x \in \mathbb{R}$ then there exists $N$ such that $N > x$.

**Corollary 2.2.** For every $\varepsilon > 0$ there is an $N$ such that $0 < 1/N < \varepsilon$.

**Lemma 2.3.** For every $x < y$, $x$ and $y$ real numbers there are integers $p$ and $q$ with $x < p/q < y$. That is, there is a rational number between $x$ and $y$.  

2
2.2 Supremums and infimums

Lemma 2.4. If \( S \) is a subset of \( \mathbb{R} \) bounded above then the least upper bound of \( S \) is unique.

Lemma 2.5. Let \( S \) be bounded above and let \( \epsilon > 0 \). Then there is some \( s \in S \) with \( s > \text{sup} \, S - \epsilon \).

Definition 2.3. The least upper bound of a set \( S \) is called the supremum of \( S \) and denoted \( \text{sup} \, S \).

We can also consider sets which are bounded below, lower bounds and greatest lower bounds. The greatest lower bound of a set \( S \) is called the infimum of \( S \) and denoted \( \text{inf} \, S \). A set which is both bounded above and bounded below is called bounded.

Proposition 2.6. Let \( S \subseteq \mathbb{R} \) be bounded below. Then \( S \) has a greatest lower bound or infimum and
\[ \text{inf} \, S = -\sup \{ -s \mid s \in S \} . \]

2.3 Limits of sequences of real numbers

Definition 2.4. A sequence of real numbers is a function \( x: \mathbb{N} \to \mathbb{R} \).

We usually write a sequence as \( \{x_i\}_{i=1}^{\infty} \) where \( x_i = x(i) \).

Definition 2.5. A sequence is called bounded if the image of the function is bounded, that is if \( \{x_1, x_2, x_3, \ldots\} \) is bounded.

Definition 2.6. A sequence \( \{x_i\}_{i=1}^{\infty} \) has a limit \( x \) if for every \( \epsilon > 0 \) there is a \( N \) such that for all \( n \geq N \) we have \( \|x_n - x\| < \epsilon \).

Note 2.1. If a sequence has a limit we say it is convergent and write \( \lim_{i \to \infty} x_i = x \) or \( x_i \to x \) as \( i \to \infty \) or just \( x_i \to x \). If a sequence does not have a limit we say that it is divergent.

Proposition 2.7. A convergent sequence is bounded.

Proposition 2.8 (Properties of limits). Let \( \{x_i\}_{i=1}^{\infty} \) and \( \{y_i\}_{i=1}^{\infty} \) be convergent sequences with \( x_i \to x \) and \( y_i \to y \) as \( i \to \infty \). Then
1. If \( x_i \to x' \) then \( x = x' \), ie limits are unique
2. If \( z_i = \lambda x_i \) for \( \lambda \in \mathbb{R} \) then \( z_i \to \lambda x \)
3. If \( z_i = x_i + y_i \) then \( z_i \to x + y \) as \( i \to \infty \).
4. If \( z_i = x_i y_i \) then \( z_i \to x \cdot y \) as \( i \to \infty \).
5. If \( y_i \neq 0 \) for all \( i \) and \( y \neq 0 \) and \( z_i = x_i / y_i \) then \( z_i \to x / y \) as \( i \to \infty \).
6. If \( x_i \leq y_i \) for all \( i \) then \( x \leq y \).

Proposition 2.9 (Sandwich Theorem). Let \( \{x_i\}_{i=1}^{\infty}, \{y_i\}_{i=1}^{\infty} \) and \( \{z_i\}_{i=1}^{\infty} \) be sequences with \( \lim_{i \to \infty} x_i = \lim_{i \to \infty} z_i = \lambda \) and such that \( x_n \leq y_n \leq z_n \) for all \( n \). Then \( \lim_{i \to \infty} y_i = \lambda \).

2.4 Cauchy sequence and completeness of the reals.

Definition 2.7. A sequence \( \{x_n\}_{n=1}^{\infty} \) of real numbers is called \textit{Cauchy} if for all \( \epsilon > 0 \) there is an \( N \) such that for all \( n, m > N \) we have \( |x_n - x_m| < \epsilon \).

Proposition 2.10. A convergent sequence is Cauchy.

Proposition 2.11. A Cauchy sequence is bounded.
2.5 Monotonic sequences

Definition 2.8. We say a sequence \( \{x_i\}_{i=1}^\infty \) is monotonically increasing if
\[
x_1 \leq x_2 \leq x_3 \leq x_4 \leq \ldots
\]
and monotonically decreasing if
\[
x_1 \geq x_2 \geq x_3 \geq x_4 \geq \ldots.
\]
A sequence which is either of these is called monotonic. If the inequalities are always strict then we say the sequence is strictly monotonic.

Theorem 2.12. A bounded monotonic sequence is convergent.

2.6 Liminf and limsup

Let \( \{x_i\}_{i=1}^\infty \) be a bounded sequence. Define \( a_n = \sup\{x_n, x_{n+1}, x_{n+2} \ldots\} \) and \( b_n = \inf\{x_n, x_{n+1}, x_{n+2} \ldots\} \).

Then we have

Lemma 2.13. The sequence \( \{a_n\}_{n=1}^\infty \) is bounded and monotonically decreasing and \( \{b_n\}_{n=1}^\infty \) is bounded and monotonically increasing.

Definition 2.9. We define the limsup of \( \{x_i\}_{i=1}^\infty \) to be \( \lim_{n \to \infty} a_n \) and the liminf of \( \{x_i\}_{i=1}^\infty \) to be \( \lim_{n \to \infty} b_n \) and denote them by \( \limsup_{n \to \infty} x_n \) and \( \liminf_{n \to \infty} x_n \) respectively.

Theorem 2.14. A sequence \( \{x_i\}_{i=1}^\infty \) converges to \( x \) if and only if it is bounded and \( \liminf_{m \to \infty} x_m = \limsup_{m \to \infty} x_m = x \).

Proposition 2.15. Every Cauchy sequence of real numbers converges.

3 Normed vector spaces and inner product spaces

Definition 3.1. A norm on a real or complex vector space \( V \) is a function \( \| \cdot \| : V \to \mathbb{R}, v \to \|v\| \) such that:
1. \( \|v\| \geq 0 \), \( \|v\| = 0 \) if and only if \( v = 0 \),
2. \( \|\lambda v\| = |\lambda|\|v\| \) for all \( v \in V \) and \( \lambda \in \mathbb{R}(\mathbb{C}) \), and
3. \( \|v + w\| \leq \|v\| + \|w\| \) for all \( v \) and \( w \) in \( V \).

A vector space with a norm is called a normed vector space.

Definition 3.2. An inner product on a real (complex) vector space \( V \) is a map \( \langle , \rangle : V \times V \to \mathbb{R}(\mathbb{C}) \), \( \langle v, w \rangle \mapsto \langle v, w \rangle \) such that for all \( v, w \) and \( u \) in \( V \) and \( \lambda \) in \( \mathbb{R}(\mathbb{C}) \), we have
1. \( \langle v, w \rangle = \langle w, v \rangle \) (hence \( \langle v, v \rangle \in \mathbb{R} \)),
2. \( \langle v, \lambda v \rangle \geq 0 \) and \( \langle v, v \rangle = 0 \) if and only if \( v = 0 \),
3. \( \langle v, w + u \rangle = \langle v, w \rangle + \langle v, u \rangle \), and
4. \( \langle v, \lambda w \rangle = \lambda \langle v, w \rangle \).

A vector space with an inner product is called an inner product space.

Lemma 3.1 (Cauchy’s inequality). If \( (V, \langle , \rangle) \) is an inner product space, and we let \( \|v\| = \sqrt{\langle v, v \rangle} \) then for all \( v \) and \( w \) in \( V \) we have
\[
\langle v, w \rangle \leq \|v\|\|w\|,
\]
with equality if and only if \( v \) and \( w \) are multiples of each other.

Lemma 3.2. If \( (V, \langle , \rangle) \) is an inner product space then the function \( \|v\| = \sqrt{\langle v, v \rangle} \) is a norm.

Definition 3.3 (Parallelogram Law). We say that the parallelogram law holds for a normed vector space \( V \) if for all \( v \) and \( w \) in \( V \) we have
\[
\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.
\]
Proposition 3.3. If $V$ is a real inner product space then the parallelogram law holds and
\[
\langle v, w \rangle = \frac{1}{4} (\|v + w\|^2 - \|v - w\|^2).
\]

Proposition 3.4. If $V$ is a real normed vector space in which the parallelogram law holds and we define
\[
\langle v, w \rangle = \frac{1}{4} (\|v + w\|^2 - \|v - w\|^2)
\]
then this is an inner product.

3.1 Norms on $\mathbb{R}^n$

Let $x = (x^1, \ldots, x^n)$ be an element of $\mathbb{R}^n$ and let $p \geq 1$. Then we define $\|x\|_p = \left(\sum_{i=1}^{n} |x^i|^p \right)^{1/p}$ and $\|x\|_\infty = \max\{|x^1|, \ldots, |x^n|\}$. We also define $\langle x, y \rangle = \sum_{i=1}^{n} x^i y^i$.

Lemma 3.5. Let $x \in \mathbb{R}^n$ and $p \geq 1$, then for any $i = 1, \ldots, n$ we have
\[
\|x_i\| \leq \|x\|_p \leq n^{1/p} \|x\|_\infty.
\]

Lemma 3.6 (Young’s inequality). If $a, b \geq 0$, $p > 1$ and $1/p + 1/q = 1$ then
\[
ab \leq \frac{a^p}{p} + \frac{b^q}{q}.
\]

Lemma 3.7 (Holder’s inequality). If $p > 1$ and $1/p + 1/q = 1$ then for any $x$ and $y$ in $\mathbb{R}^n$ we have
\[
|\langle x, y \rangle| \leq \|x\|_p \|y\|_q.
\]

Lemma 3.8 (Minkowski’s inequality). If $p > 1$ then
\[
\|x + y\|_p \leq \|x\|_p + \|x\|_q.
\]

Proposition 3.9. The functions $\| \cdot \|_p$ for $p \geq 1$ or $p = \infty$ are norms on $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is an inner product.

3.2 Convexity and normed spaces

Definition 3.4. Let $v$ and $w$ be elements of a vector space $V$ then we define
\[
[v, w] = \{(1 - t)v + tw \mid t \in [0, 1]\}.
\]

Definition 3.5. If $C \subseteq V$ a vector space we say that $C$ is convex if for every $v, w \in C$ we have $[v, w] \subseteq C$.

Proposition 3.10. If $V$ is a normed vector space then the set
\[
S = \{v \in V \mid \|v\| \leq 1\}
\]
is convex.

3.3 Orthonormal sets

Definition 3.6. A subset $E \subseteq V$ of a vector space is called linearly independent if there are no $v_1, v_2, \ldots, v_r \in E$ with $c_1 v_1 + c_2 v_2 + \ldots + c_r v_r = 0$ except when $c_1 = c_2 = \ldots c_r = 0$.

Definition 3.7. A subset $E = \{e_1, e_2, \ldots\}$ of an inner product space $V$ is called a (countable) orthonormal set if $\langle e_i, e_j \rangle = \delta_{ij}$, i.e. it is zero unless $i = j$ when it is 1.

Lemma 3.11. An orthonormal set is linearly independent.

Lemma 3.12. Let $v_1, v_2, \ldots$ be a countable linearly independent set in an inner product space $V$. Then we can use Gram-Schmidt orthogonalisation to produce an orthonormal set.
3.4 Convergence in normed vector spaces

**Definition 3.8.** A sequence in a normed vector space $V$ is a function $x: \mathbb{N} \rightarrow V$.

We usually write a sequence as $\{x_i\}_{i=1}^\infty$ where $x_i = x(i)$.

**Definition 3.9.** A sequence $\{x_i\}_{i=1}^\infty$ in a normed vector space $V$ has a limit $x$ if for every $\epsilon > 0$ there is an $N$ such that for all $n \geq N$ we have

$$\|x_n - x\| \leq \epsilon.$$ 

**Note 3.1.** If a sequence has a limit we say it is convergent and write $\lim_{i \rightarrow \infty} x_i = x$ or $x_i \rightarrow x$ as $i \rightarrow \infty$ or just $x_i \rightarrow x$. If a sequence does not have a limit we say that it is divergent.

**Proposition 3.13.** Let $\{x_i\}_{i=1}^\infty$ be a sequence in a normed vector space $V$ then the following are equivalent:

1. $\{x_i\}_{i=1}^\infty$ converges to $x$
2. $\{x_i - x\}_{i=1}^\infty$ converges to $0$
3. $\{\|x_i - x\|\}_{i=1}^\infty$ converges to $0$.

**Proposition 3.14 (Properties of limits).** Let $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ be convergent sequences with $x_i \rightarrow x$ and $y_i \rightarrow y$ as $i \rightarrow \infty$. Then

1. If $x_i \rightarrow x'$ then $x = x'$, i.e., limits are unique.
2. If $z_i = x_i + y_i$ then $z_i \rightarrow x + y$ as $i \rightarrow \infty$.
3. If $\lambda \in \mathbb{R}$ and $z_i = \lambda x_i$ then $z_i \rightarrow \lambda x$.

**Definition 3.10.** A sequence $\{x_n\}_{n=1}^\infty$ of vectors in a normed vector space is called Cauchy if for all $\epsilon > 0$ there is an $N$ such that for all $n, m > N$ we have $\|x_n - x_m\| < \epsilon$.

**Proposition 3.15.** A convergent sequence in a normed vector space is Cauchy.

**Definition 3.11.** A normed vector space is called complete if every Cauchy sequence converges.

**Definition 3.12.** A complete normed vector space is called a Banach space and a complete normed inner product space is called a Hilbert space.

3.5 Equivalent norms

**Definition 3.13.** Two norms $\|\|_1$ and $\|\|_2$ are called equivalent if there are constants $c$ and $C$ such that for all $v \in V$ we have

$$c\|v\|_1 \leq \|v\|_2 \leq C\|v\|_1.$$

**Proposition 3.16.** Let $V$ be a vector space with two norms $\|\|_1$ and $\|\|_2$ which are equivalent. Then:

1. A sequence $\{v_n\}_{n=1}^\infty$ is Cauchy in $\|\|_1$ if and only if it is Cauchy in $\|\|_2$.
2. $v_n \rightarrow v$ in $\|\|_1$ if and only if $v_n \rightarrow v$ in $\|\|_2$.

**Proposition 3.17.** Let $V$ be a vector space with two norms $\|\|_1$ and $\|\|_2$ which are equivalent. Then $V$ is complete for $\|\|_1$ if and only it is complete for $\|\|_2$.

**Proposition 3.18.** Any two of the norms $\|\|_p$ for $p \geq 1$ or $p = \infty$ on $\mathbb{R}^n$ are equivalent.

**Proposition 3.19.** The normed vector space $(\mathbb{R}^n, \|\|_p)$ is complete for any $p \geq 1$ or $p = \infty$.

4 Metric spaces

**Definition 4.1.** A metric on a set $X$ is a function $d: X \times X \rightarrow \mathbb{R}$ such that

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$ (positivity)
2. $d(x, y) = d(y, x)$ (symmetry)
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

for all $x, y$ and $z \in X$.

**Definition 4.2.** A metric space is a pair $(X, d)$ where $d$ is a metric on a set $X$.

**Lemma 4.1.** If $(V, \|\|)$ is a normed vector space then $d(v, w) = \|v - w\|$ is a metric.

**Definition 4.3 (Subspace metric).** If $(X, d)$ is a metric space and $Y \subseteq X$ we define $d_Y(y_1, y_2) = d(y_1, y_2)$. This is a metric on $Y$ called the subspace metric.
4.1 Open sets

Definition 4.4. Let \((X, d)\) be a metric space, \(x \in X\) and \(\delta > 0\). Then we define

\[
B(x, \delta) = \{ y \in X \mid d(x, y) < \delta \}
\]

the open ball around \(x\) of radius \(\delta\).

Definition 4.5. If \((X, d)\) is a metric space a subset \(U \subseteq X\) is called open if \(\forall x \in U\) there is a \(\delta > 0\) such that \(B(x, \delta) \subseteq U\).

Proposition 4.2. If \((X, d)\) is a metric space, \(x \in X\) and \(\delta > 0\) then \(B(x, \delta)\) is open.

Proposition 4.3. A subset of a metric space is open if and only it is a union of open balls.

Theorem 4.4. Let \((X, d)\) be a metric space then

1. \(X\) and \(\emptyset\) are open;
2. If \(U_\alpha\) is open for every \(\alpha \in I\) then \(\bigcup_{\alpha \in I} U_\alpha\) is open, and
3. If \(U_i\) is open for every \(i = 1, \ldots, n\) then \(\bigcap_{i=1}^n U_i\) is open.

Definition 4.6. If \(x\) is an element of a metric space \(X\) and \(N \subseteq X\) we call \(N\) a neighbourhood of \(x\) if \(\exists \delta > 0\) such that \(B(x, \delta) \subseteq N\).

Lemma 4.5. A set is open if and only it is a neighbourhood of all its points.

4.2 Closed sets

Definition 4.7. A subset \(C\) of a metric space \(X\) is called closed if \(C^c\) is open.

Proposition 4.6. The closed ball 

\[
\overline{B}(x, \delta) = \{ y \in X \mid d(x, y) \leq \delta \}
\]

is closed.

Theorem 4.7. Let \((X, d)\) be a metric space then

1. \(X\) and \(\emptyset\) are closed;
2. If \(C_\alpha\) is closed for every \(\alpha \in I\) then \(\bigcap_{\alpha \in I} C_\alpha\) is closed, and
3. If \(C_i\) is closed for every \(i = 1, \ldots, n\) then \(\bigcup_{i=1}^n C_i\) is closed.

Definition 4.8. If \(A\) is a subset of a metric space \(X\) then we define the closure of \(A\) by

\[
\overline{A} = \{ x \in X \mid B(x, \delta) \cap A \neq \emptyset, \forall \delta > 0 \}.
\]

Proposition 4.8. Let \(A\) be a subset of a metric space \(X\) then:

1. \(A \subseteq \overline{A}\);
2. if \(C\) is a closed set containing \(A\) then \(\overline{A} \subseteq C\); and
3. \(\overline{A}\) is the intersection of all the closed sets containing \(A\).
4. \(A\) is closed if and only if \(A = \overline{A}\).

Definition 4.9. A subset \(A\) of a metric space \(X\) is called dense if \(\overline{A} = X\).

Definition 4.10. A metric space is called separable if it has a countable dense subset.
4.3 Convergence in metric spaces

Definition 4.11. Let \( \{x_n\}_{n=1}^\infty \) be a sequence in a metric space \( X \). We say that \( \{x_n\}_{n=1}^\infty \) converges to a point \( x \) if for all \( \epsilon > 0 \), \( \exists N \) such that for all \( n > N \) we have \( d(x_n, x) < \epsilon \). In that case we write \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) as \( n \to \infty \) and say that \( \{x_n\}_{n=1}^\infty \) is convergent.

Lemma 4.9. A sequence \( \{x_n\}_{n=1}^\infty \) converges to \( x \) if and only if \( \lim_{n \to \infty} d(x_n, x) = 0 \).

Lemma 4.10. A sequence in a metric space has, at most, one limit.

Definition 4.12. Let \( A \) be a subset of a metric space \( X \). A point \( x \in X \) is called a limit point of \( A \) if there is a sequence \( \{x_n\}_{n=1}^\infty \subseteq A \setminus \{x\} \) converging to \( x \).

Proposition 4.11. If \( A \) is a subset of a metric space then the closure of \( A \) is the union of \( A \) with all its limit points.

Corollary 4.12. A set \( A \) is closed if and only if it contains all its limit points.

Definition 4.13. A sequence \( \{x_n\}_{n=1}^\infty \) in a metric space \( X \) is called Cauchy if for all \( \epsilon > 0 \) there is an \( N \) such that for all \( n, m > N \) we have \( d(x_n, x_m) < \epsilon \).

Lemma 4.13. A convergent sequence is Cauchy.

Definition 4.14. A metric space \((X, d)\) is called complete if every Cauchy sequence in \( X \) converges.

Proposition 4.14. Suppose that \( Y \) is a subset of a metric space \( X \) and \((Y, d_Y)\) is complete then \( Y \) is closed.

Proposition 4.15. Suppose that \((X, d)\) is complete and \( Y \subseteq X \) is closed then \((Y, d_Y)\) is complete.

4.4 Completeness of \( C[a, b] \)

Definition 4.15. If \( f : [a, b] \to \mathbb{R} \) we say that \( f \) is continuous at \( x \in [a, b] \) if for every \( x_n \to x \) we have \( f(x_n) \to f(x) \).

Note 4.1. Below in Section ?? we give a more general definition of continuity and Proposition ?? shows that for \([a, b]\) it is equivalent to this one.

Definition 4.16. A function \( f \) defined on \([a, b]\) is called bounded if there is an \( R > 0 \) such that for all \( x \in [a, b] \) we have \(|f(x)| < R\).

Proposition 4.16. A continuous function on \([a, b]\) is bounded.

Note 4.2. Compare this to Proposition ?? where we show that a continuous function on a compact set is bounded.

If \( f \) is bounded but perhaps not continuous we can still define

\[
\|f\|_{\infty} = \sup\{|f(x)| \mid x \in [a, b]\}.
\]

Definition 4.17. Let \( \{f_n\}_{n=1}^\infty \) be a sequence of functions \( f_n : [a, b] \to \mathbb{R} \). We say that \( \{f_n\}_{n=1}^\infty \) converges pointwise to a function \( f : [a, b] \to \mathbb{R} \) if \( \lim_{n \to \infty} f_n(x) = f(x) \) for every \( x \in [a, b] \).

Definition 4.18. If \( \{f_n\}_{n=1}^\infty \) is a sequence in \( C[a, b] \) which converges to \( f \) in the uniform norm then \( \{f_n\}_{n=1}^\infty \) converges to \( f \) pointwise.

Lemma 4.17. If \( \{f_n\}_{n=1}^\infty \) is a sequence in \( C[a, b] \) and \( f : [a, b] \to \mathbb{R} \) such that

1. \( f \) is bounded, and
2. \( \|f_n - f\|_{\infty} \to 0 \) as \( n \to \infty \)

then \( f \) is continuous.

Theorem 4.18. The metric space \( C[a, b] \) with the uniform metric \( d_\infty \) induced by the uniform norm

\[
\|f\|_{\infty} = \sup\{|f(x)| \mid x \in [a, b]\}
\]

is complete.
4.5 Fixed points and the contraction mapping theorem

Definition 4.19. If $T: X \to X$ is a map from a set $X$ to itself a fixed point of $T$ is an $x \in X$ such that $T(x) = x$.

Definition 4.20. Let $(X, d)$ be a metric space and $T: X \to X$. We call $T$ a contraction if there is a $K$, $0 < K < 1$ such that $d(T(x), T(y)) \leq Kd(x, y)$ for all $x, y \in X$.

Theorem 4.19 (Contraction Mapping Theorem). If $(X, d)$ is a complete metric space and $T: X \to X$ is a contraction then $T$ has a unique fixed point.

Theorem 4.20 (Picard’s Theorem). If $f: \mathbb{R} \to \mathbb{R}$ satisfies $\exists M$ such that $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| < M|x - y|$ then $\exists a > 0$ and a unique $y \in C[0, a]$ such that

$$y(0) = 0$$

$$\frac{dy}{dt} = f(y(t)).$$

4.6 Continuous functions

Definition 4.21. Let $(X, d)$ and $(Y, d)$ be metric spaces. A function $f: X \to Y$ is said to be continuous at $x \in X$ if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x, x' \in X$ with $d(x, x') < \delta$ we have $d(f(x), f(x')) < \epsilon$.

Lemma 4.21. A function $f: X \to Y$ between metric spaces is continuous at $x \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \epsilon)$.

Proposition 4.22. A function $f: X \to Y$ between metric spaces is continuous at $x \in X$ if and only if for every sequence $x_n \to x$ we have $f(x_n) \to f(x)$.

Definition 4.22. A function $f: X \to Y$ between metric spaces is called continuous if it is continuous at $x$ for every $x \in X$.

Proposition 4.23. A function $f: X \to Y$ between metric spaces is continuous if and only if for every open set $U \subseteq Y$ we have that $f^{-1}(U) \subseteq X$ is open.

Corollary 4.24. A function $f: X \to Y$ between metric spaces is continuous if and only if for every closed set $C \subseteq Y$ we have that $f^{-1}(C) \subseteq X$ is closed.

Proposition 4.25. If $T: V \to W$ is a linear map between normed vector spaces then the following are equivalent:

1. $T$ is continuous,
2. $T$ is continuous at $0$,
3. $T$ is continuous at some $v \in V$,
4. there is a constant $C$ such that for all $v \in V$ we have $\|T(v)\| \leq C\|v\|$, and
5. there is a constant $C$ such that for all $v \in V$ with $\|v\| = 1$ we have $\|T(v)\| \leq C$.

Note 4.3. A continuous linear function between normed vector spaces is sometimes called bounded.

Definition 4.23. Let $T: V \to W$ be continuous and linear. Define $\|T\| = \sup\{\|T(v)\| \mid v \in V, \|v\| = 1\}$.

Lemma 4.26. If $T: V \to W$ is linear and continuous then $\|T(v)\| \leq \|T\|\|v\|$ for all $v \in V$.

Proposition 4.27. The function $\|T\|$ on continuous linear maps $T: V \to W$ is a norm and makes the set of all continuous linear maps from $V$ to $W$ a normed vector space.
5 Compactness

Definition 5.1. If $A$ is a subset of a metric space $X$ then an open covering of $A$ is a collection $\{U_\alpha\}_{\alpha \in I}$ of open subsets of $X$ such that $A \subseteq \bigcup_{\alpha \in I} U_\alpha$.

Definition 5.2. If $\{U_\alpha\}_{\alpha \in I}$ is an open cover of $A \subseteq X$ then a subcover is a collection $\{U_\alpha\}_{\alpha \in J}$ for some $J \subseteq I$ which is still a cover of $A$.

We call a subcover finite if it contains a finite number of open sets, i.e. $J$ is finite.

Definition 5.3 (Compactness). A subset $A$ of a metric space $X$ is called compact if every open cover of $A$ has a finite subcover.

Theorem 5.1. A compact subset of a metric space is closed.

Definition 5.4 (Bounded subset). A subset $A$ of a metric space $X$ is called bounded if there is an $x \in X$ and an $R$ such that $A \subseteq B(x, R)$.

Theorem 5.2. A compact subset of a metric space is bounded.

Theorem 5.3. A closed subset of a metric space is compact.

Definition 5.5 (Subsequence). Let $\{x_i\}_{i=1}^\infty$ be a sequence in a metric space $(X, d)$. Let $\phi: \mathbb{N} \to \mathbb{N}$ be an order preserving map, that is $i < j \implies \phi(i) < \phi(j)$ for all $i$ and $j$. The sequence $\{y_j\}_{j=1}^\infty$ defined by $y_j = x_{\phi(j)}$ is called a subsequence of $\{x_i\}_{i=1}^\infty$.

Note 5.1. Usually we do not bother with making $\phi$ explicit but write $y_1 = x_{i_1}, y_2 = x_{i_2}, \ldots$ and say that $\{x_{i_j}\}_{j=1}^\infty$ is a subsequence of $\{x_i\}_{i=1}^\infty$.

Lemma 5.4. If $\{x_i\}_{i=1}^\infty$ is a convergent sequence with $\lim_{i \to \infty} x_i = x$ then for any subsequence $\{x_{i_j}\}_{j=1}^\infty$ we have $\lim_{j \to \infty} x_{i_j} = x$.

Definition 5.6 (Sequential compactness). A subset $A$ of a metric space $X$ is called sequentially compact if every sequence in $A$ has a convergent subsequence.

Theorem 5.5. A compact subset of a metric space is sequentially compact.

Corollary 5.6. A closed metric space is complete.

Lemma 5.7 (Lebesgue covering lemma). Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of a sequentially compact subset $A$ of a metric space $X$. Then there is a $\delta > 0$ such that for every $a \in A$ there is an $\alpha \in I$ such that $B(a, \delta) \subseteq U_\alpha$.

Theorem 5.8 (Bolzano-Weierstrass). A sequentially compact subset of a metric space is compact.

Theorem 5.9 (Heine-Borel). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

5.1 Compactness and continuity

Proposition 5.10. If $f: X \to Y$ is a continuous function between metric spaces and $A \subseteq X$ is compact then $f(A)$ is compact.

Proposition 5.11. If $f: X \to \mathbb{R}$ is continuous and $X$ is compact then $f$ is bounded and attains its inf and sup.

Definition 5.7. If $(X, d)$ is a compact metric space we denote by $C(X)$ the vector space of all continuous functions on $X$ with the uniform metric $d_\infty$ induced by the uniform norm

$$||f||_\infty = \sup_{x \in X} |f(x)|.$$

Note 5.2. Just as for $C[a, b]$ the space $C(X)$ is complete if $X$ is compact.

Definition 5.8 (Uniform continuity). A function $f: X \to Y$ between metric spaces if called uniformly continuous if $\forall \epsilon > 0$ there is a $\delta > 0$ such that for all $x, x' \in X$ we have that $d(f(x), f(x')) < \epsilon$ whenever $d(x, x') < \delta$.

Proposition 5.12. If $X$ is compact and $f: X \to Y$ is continuous then $f$ is uniformly continuous.
Proposition 5.13. Let \( \phi : [a, b] \to \mathbb{R} \) and \( k : [a, b] \times [a, b] \to \mathbb{R} \) be continuous. Let \( M \) be the supremum of \( |k(s, t)| \) for \( (s, t) \in [a, b] \times [a, b] \). Then if \( |\mu||b - a|M < 1 \) there is a unique continuous solution to the integral equation
\[
x(t) = \phi(t) + \mu \int_a^b k(s, t)x(s)ds.
\]

Definition 5.9 (Equicontinuity). A subset \( A \subseteq C(X) \), of the space of all continuous real valued functions on a metric space \( X \), is called equicontinuous if \( \forall \epsilon > 0 \) there is a \( \delta > 0 \) such that for all \( f \in A \) and for all \( x, x' \in X \) with \( d(x, x') < \delta \) we have \( |f(x) - f(x')| < \epsilon \).

Theorem 5.14 (Arzela-Ascoli). Let \( (X, d) \) be a compact metric space. A subset \( A \subseteq C(X) \) is compact if and only if it is closed, bounded and equicontinuous.

Proposition 5.15. Let \( (X, d) \) be a metric space and \( Y \subseteq X \) a subset with the subspace metric \( d_Y \). Then \( U \subseteq Y \) is open in \( (Y, d_Y) \) if and only if \( U = Y \cap V \) for \( V \subseteq X \) open in \( (X, d) \).

Proposition 5.16. Let \( (X, d) \) be a metric space. Then \( A \) is a compact subset of \( X \) if and only if \( (A, d_A) \) is a compact metric space.

6 Topological spaces

Definition 6.1. Let \( X \) be a set and \( \mathcal{P}(X) \) the power set of \( X \), that is the collection of all subsets of \( X \). Then a topology on \( X \) is a subset \( \mathcal{T} \subseteq \mathcal{P}(X) \) such that:
(i) \( X \) and \( \emptyset \) are in \( \mathcal{T} \)
(ii) If \( U_\alpha \in \mathcal{T} \) for every \( \alpha \in I \) then \( \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T} \)
(iii) If \( U_i \in \mathcal{T} \) for every \( i = 1, \ldots, n \) then \( U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T} \).

If \( \mathcal{T} \) is a topology we call the sets in \( \mathcal{T} \) open and a set \( X \) with a topology \( \mathcal{T} \) is called a topological space. A subset \( A \subseteq X \) is called closed if \( A^c \) is open.

Proposition 6.1. If \( (X, d) \) is a metric space then the open sets form a topology for \( X \). We call this the metric topology.

Definition 6.2. A topological space \( X \) is called Hausdorff if for every \( x, y, \in X \) there are open sets \( U \) and \( V \) with \( x \in U, y \in V \) and \( U \cap V = \emptyset \).

Lemma 6.2. A metric space is Hausdorff.

Definition 6.3. A function \( f : X \to Y \) between topological spaces is called continuous if for every open set \( U \subseteq Y \) we have \( f^{-1}(U) \subseteq X \) open.

Definition 6.4. If \( X \) is a topological space then a subset \( A \subseteq X \) is called compact if every open cover of \( A \) has a finite subcover.

Note 6.1. The definition of open cover and finite subcover are the same as in the metric space case in Section ??.

Proposition 6.3. If \( V \) is a vector space with equivalent norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) then the metrics they define both give rise to the same topology.

Proposition 6.4. If \( V \) is a finite dimensional vector space then any two norms on \( V \) are equivalent.

Proposition 6.5. If \( V \) is a finite dimensional vector space with two norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) then a function \( f : V \to X \) a topological space is continuous in \( \| \cdot \|_1 \) if and only if it is continuous in \( \| \cdot \|_2 \).

Proposition 6.6. If \( V \) is a finite dimensional vector space with two norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) then a subset \( A \subseteq V \) is compact in \( \| \cdot \|_1 \) if and only if it is compact in \( \| \cdot \|_2 \).
6.1 Connectedness

**Proposition 6.7.** Let \((X, d)\) be a metric space and \(Y \subseteq X\) a subspace with the subspace metric. Then \(U \subseteq Y\) is open if and only if \(U = W \cap Y\) for some \(W \subseteq X\) open.

**Definition 6.5.** A topological space \(X\) is called disconnected if there are open sets \(U\) and \(V\) in \(X\) such that \(X = U \cup V\), \(U \neq \emptyset \neq V\) and \(U \cap V = \emptyset\).

**Definition 6.6.** Let \(X\) be a topological space. A subset \(A \subseteq X\) is called disconnected if there are open sets \(U\) and \(V\) in \(X\) such that \(A \subseteq U \cup V\), \(U \cap A \neq \emptyset \neq V \cap A\) and \(U \cap V \cap A = \emptyset\).

**Definition 6.7 (Connectedness).** A subset \(A\) of a topological space is called connected if it is not disconnected.

**Proposition 6.8.** A subset of the real line is connected if and only if it is an interval.

**Theorem 6.9.** If \(f : X \to Y\) is a continuous map between metric spaces and \(X\) is connected then \(f(X)\) is connected.

7 Hilbert space

**Definition 7.1.** We say that a sequence of vectors \(\{x_n\}_{n=1}^{\infty}\) in a normed vector space has an infinite sum if \(\lim_{r \to \infty} \sum_{i=1}^{r} x_i\) exists. If this limit exists we denote it by \(\sum_{i=1}^{\infty} x_i\). We often just say that \(\sum_{i=1}^{\infty} x_i\) exists, or that the infinite series \(\sum_{i=1}^{\infty} x_i\) converges, to mean that \(\{x_n\}_{n=1}^{\infty}\) has an infinite sum.

We denote by \(\ell_2\) (called little ell two) the set of all sequences \(x = \{x_n\}_{n=1}^{\infty}\) of real numbers for which \(\sum_{i=1}^{\infty} |x_i|^2\) exists. If \(x, y \in \ell_2\) then it follows from Cauchy’s inequality that \(\sum_{n=1}^{\infty} x_n y_n\) exists and we denote this by \(\langle x, y \rangle\). The space \(\ell_2\) with this inner product is a Hilbert space.

**Theorem 7.1.** Let \(C\) be a closed, nonempty, convex subset of a Hilbert space. Then \(C\) has a unique point of smallest norm.

**Definition 7.2.** We say two points \(v\) and \(w\) in an inner product space \(V\) are orthogonal or perpendicular if \(\langle v, w \rangle = 0\).

**Definition 7.3.** Let \(A\) be a subset of an inner product space \(H\) then we define the perpendicular or orthogonal space to \(A\) by

\[ A^\perp = \{ v \in V \mid \langle v, a \rangle = 0, \forall a \in A\} \]

**Proposition 7.2.** Let \(A\) be a subset of an inner product space \(V\).

1. \(A^\perp\) is a closed subspace of \(V\),
2. \(A \subseteq A^\perp\),
3. \(A \subseteq B \Rightarrow B^\perp \subseteq A^\perp\), and
4. \(A^\perp \cap A = \{0\}\) or \(\emptyset\).

**Proposition 7.3.** If \(A \subseteq V\) is a closed subspace of a Hilbert space \(H\) and \(v \in H\) then there exists a unique \(a_0 \in A\) such that \(x - a_0 \in A^\perp\). Moreover

\[ \|x - a_0\| = \inf\{\|x - a\| \mid a \in A\}. \]

**Corollary 7.4.** Let \(A\) be a closed subspace of a Hilbert space \(H\). Then

1. \(H = A \oplus A^\perp\), and
2. \(A = A^\perp\perp\).

**Proposition 7.5.** Let \(A\) be a non-empty closed subspace of Hilbert space \(H\). Define \(P : H \to H\) by \(v = P(v) + w\) where \(w \in A^\perp\) and \(P(v) \in A\). We have

1. \(P\) is linear and continuous and \(\|P\| = 1\),
2. \(P(a) = a\) for all \(a \in A\),

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3. the image of $P$ is $A$,
4. the kernel of $P$ is $A^\perp$, and
5. $P^2 = P$.

Note 7.1. The map $P$ is called the orthogonal projection of $H$ onto $A$.

**Proposition 7.6.** Let $H$ be a real Hilbert space and let $v \in H$. Define $L_v : H \to \mathbb{R}$ by $L_v(w) = \langle v, w \rangle$ then $L_v$ is linear and continuous and $\|L_v\| = \|v\|$.

**Definition 7.4 (Dual space).** If $V$ is a normed vector space denote by $V^*$ the space of all linear continuous functions from $V$ to $\mathbb{R}$.

**Theorem 7.7 (Riesz Representation).** Let $H$ be a real Hilbert space. The map $\phi : H \to H^*$ defined by $\phi(v) = L_v$ is linear, one-to-one and onto.

Note 7.2. If $H$ is a complex Hilbert space the same result holds but $\phi$ is conjugate linear in the sense that $\phi(\lambda v) = \overline{\lambda} \phi(v)$.

**Proposition 7.8 (Adjoint of a continuous linear map).** Let $T : H \to H$ be linear and continuous. Then there is a map $T^* : H \to H$ determined uniquely by the condition that for all $v, w, \in H$ we have

$$\langle T^*(v), w \rangle = \langle v, T(w) \rangle.$$  

This map $T^*$ is linear and continuous and $\|T^*\| = \|T\|$. We call $T^*$ the adjoint of $T$.

**Proposition 7.9.** If $T : H \to H$ is linear and continuous then $\ker(T) = \text{im}(T^*)^\perp$ and $\ker(T^*) = \text{im}(T)^\perp$.  