A few comments on Assignments 3 and 4.

Assignment 3:

A number of people had a better proof than me for Question 2. Instead of $\epsilon$ and $\delta$ you can just apply the sandwich theorem and take the limit as $n \to \infty$ for $0 \leq \|v_n - w\| = \|v_n - v\| \leq \|v_n - w\|$. This avoids worrying about $\|w\| = 0$.

From the definition of $\|f\|_\infty = \sup \{|f(x)| | x \in [0, 1]\}$ we know that $\|f\|_\infty$ is an upper bound for the set $\{|f(x)| | x \in [0, 1]\}$ and immediately we have $|f(x)| \leq \|f\|_\infty$ for any $x \in [0, 1]$.

If $x = 1$ then $\lim_{n \to \infty} x^n = 1$ and if $x \in (0, 1)$ then $\lim x^n \to 0$. There is nothing diverging.

Don’t forget to show that if $d(x, y) = 0$ then $x = y$ when you are proving something is a metric.

In the last question some people had the correct idea but had trouble writing it down. If you say ‘So for all $\epsilon > 0$ the only way to have $d(x_n, x_m) \leq \epsilon$ is to have $d(x_n, x_m) = 0,’$ that’s not correct as written as for $\epsilon = 0$ there is no problem satisfying this condition if $x_n \neq x_m$. It needs careful writing out. I think the easiest thing is to pick an $\epsilon < 1$ rather than worrying about all of them.

Assignment 4:

If $C_j$ is a column of $C$ some people tried to use Cauchy on this. This won’t work. You can actually use Holder for $p = 1, q = \infty$ but we did not prove that case.

Thanks to the person who pointed out that you can use $\|Cx\|_\infty \leq \|C\|_\infty \|x\|_\infty$ where

$$\|C\|_\infty = \sup \{|Cx|_\infty | \|x\|_\infty = 1\}.$$  

You do have to calculate this though and show it is what you want it to be. You can do this as follows.

$$\|Cx\|_\infty = \max\{|\sum_{k=1}^{n} C_{1k}x_k|, \ldots, |\sum_{k=1}^{n} C_{nk}x_k|\}. $$

so it suffices to look at one of these say $|\sum_{k=1}^{n} C_{ik}x_k|$. Notice that we can change the sign of any of the entries in $x$ without changing the condition that $\|x\|_\infty = 1$ so at the max each $C_{ik}x_k$ is non-negative. In this case the max of this sum will occur which each $x_k = \pm 1$ and will be $\sum_{k=1}^{n} |C_{ik}|$.

Hence

$$\|Cx\|_\infty = \max\{|\sum_{k=1}^{n} C_{1k}|, \ldots, |\sum_{k=1}^{n} C_{nk}|\}$$

as required.

In the definition of contraction given in class I should have allowed $K = 0$. Of course if $K = 0$ then $d(T(x), T(y)) = 0$ so the map is constant and a fixed point is immediate. It is best to change the definition I gave in my lectures and the handouts. But obviously I will take either definition in the exam.

In the bit where you have to prove $x_j \geq 0$ some people got this directly which is neat. You have $0 \leq d_j = x_j - \sum C_{jk}x_k$ for all $j$. Pick $i$ so that $x_i$ is smallest of all the $x_k$ then $\sum C_{jk}x_k \geq \sum C_{jk}x_i$ so $0 \leq x_i - \sum C_{jk}x_k \leq x_j - \sum C_{jk}x_i = (1 - \sum C_{jk})x_i$. But $C_{jk} \geq 0$ so $\sum C_{jk} < 1$ so $x_i \geq 0$. But $x_i$ is the smallest so they are all non-negative.