1. Use the $n$-th term test to show that $\sum_{n=0}^{\infty} \frac{n+2}{n+3}$ diverges.

2. Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Using the fact that $\frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}$ show that

$$\sum_{n=1}^{k} \frac{1}{n(n+1)} = 1 - \frac{1}{k+1}$$

Hence show that $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

3. (a) Using the fact that $n^2 \geq n(n-1)$ and the previous question show that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by applying the Comparison Test.

(b) Use the Comparison Test to show that $\sum_{n=1}^{\infty} \frac{a^n}{n^{2n+3}}$ converges for any $a \in \mathbb{R}$.

4. Prove that for every $z \in \mathbb{C}$ we have $\text{Im}(z) \leq |z|$. For which $z \in \mathbb{C}$ is this an equality.

5. Find $\text{Re}(z)$, $\text{Im}(z)$ and $|z|$ if:

(a) $z = \sqrt{2} - \sqrt{2}i$
(b) $\frac{2+3i}{i+6i}$
(c) $z = 2 \text{cis}(\frac{\pi}{4})$.

6. Write the following complex numbers in the form $r \text{cis}(\theta)$:

(a) $\sqrt{2} - \sqrt{2}i$
(b) $\frac{2}{1-i}$
(c) $(1-i)^3$.

7. The quaternions $\mathbb{H}$ were invented by the mathematician Sir William Rowan Hamilton\(^1\). They are a four-dimensional vector space. A quaternion looks like $q = t + xi + yj + zk$ and the multiplication is defined by requiring that $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$ and $ijk = -1$. So, for example $(1 + i)(j + 2k) = j + 2k + ij + 2ik = j + 2k + k - 2j = -j + 3k$. We define the quaternion conjugate by $\overline{q} = t - xi - yj - zk$. We call a quaternion real if $x = y = z = 0$.

(a) Show that $(1+i)(j+2k) \neq (j+2k)(1+i)$ (i.e quaternion multiplication is not commutative).

(b) Show that $\overline{pq} = \overline{qp}$ for any quaternions $p$ and $q$.

(c) Show that $q\overline{q}$ is real and calculate it.

(d) Show that if $q \neq 0$ then there is a quaternion $q^{-1}$ such that $qq^{-1} = q^{-1}q = 1$.

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\(^1\)http://www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Hamilton.html