PURE MTH 2005 Real & Complex Analysis II

Tutorial Exercise 1.

Handed out Thursday 7th August. Please try the problems before your tutorial.

1. Define a subset $S \subset \mathbb{Q}$ to be a *cut* if (a) S is bounded above, (b) S does not contain its upper bound, (c) if $s \in S$ and $t < s$ then $t \in S$. Show that if $r \in \mathbb{Q}$ then the set

$$S_r = \{t \in \mathbb{Q} \mid t < r\}$$

is a cut.

2. Which of the following sequences are bounded above or below ?

(a) $x_n = 1/n^2$ (b) $x_n = -n^2 + 2n$ (c) $x_n = \sin(n^2 - n)$

3. Consider the sequence $(x_n)_{n=1}^{\infty}$ where

$$x_n = \frac{3n+5}{n+3}.$$

Find the limit L of this sequence and use the definition of the limit of a sequence to prove that $\lim_{n\to\infty} x_n = L$.

4. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences with $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Define a sequence $(c_n)_{n=1}^{\infty}$ by $c_n = 3a_n - b_n$ for all $n \in \mathbb{N}$ and show that $\lim_{n\to\infty} c_n = 3A - B$. Use the definition of limit not the limit laws.

5. Suppose that $(a_n)_{n=1}^{\infty}$ is a sequence with $\lim_{n\to\infty} a_n = A$. Let $c \in \mathbb{R}$ and define $b_n = ca_n$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} b_n = cA$. Be careful with the case $c = 0$.

6. Supposed that $(a_n)_{n=1}^{\infty}$ is a bounded sequence and that $(b_n)_{n=1}^{\infty}$ is a sequence which converges to 0. Show that $\lim_{n\to\infty} a_n b_n = 0$.

7. Let $(x_n)_{n=1}^{\infty}$ be a sequence converging to $L > 0$. Show that there is an N such that for all $n \geq N$ we have $x_n > 0$.

8. Let $(a_n)_{n=1}^{\infty}$ be a sequence and L be a real number. Give a formal definition of what it means for the sequence *not* to converge to L.

9. Give an example of:

(a) A bounded sequence which is not convergent;

(b) A convergent sequence which is not monotonic;

(c) A monotonic sequence which is not convergent.

10*. If S and T are cuts (see Question 1) show that $S + T = \{s + t \mid s \in S, t \in T\}$ is a cut.

Cuts were invented by the mathematician Dedekind (1831–1916) and are usually called Dedekind cuts. He used them to define real numbers. Each cut corresponds to a real number. He showed that you could add and multiply cuts and that they satisfied all the axioms of the real numbers including the least upper bound axiom. This question shows how to add cuts. Think about how you would define the product of two cuts, the negative of a cut, the reciprocal of a cut and when one cut is greater than another.

I am happy to discuss this with anyone or there are lots of books in the library on Real Analysis that will talk about Dedekind cuts. For information about Dedekind himself see `http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Dedekind.html`.