1. (a) Let \( z = x + iy \) then \( \text{Re}(z) = x \leq \sqrt{x^2 + y^2} = |z| \). (b) Either let \( z = |z| \text{cis} (\theta) \) so that \( z^{-1} = (1/|z|) \text{cis} (-\theta) \) and \( |z^{-1}| = 1/|z| \) or let \( z = x + iy \) so that \( z^{-1} = (x - iy)/(x^2 + y^2) \) and \( |z^{-1}| = \sqrt{x^2/(x^2 + y^2)^2 + y^2/(x^2 + y^2)^2} = 1/\sqrt{x^2 + y^2} \).

\[ \text{[a=1, b=2]} \]

2. (a) \( \text{Re}(z) = 4, \text{Im}(z) = -3 \) and \( |z| = 5 \). (b) \( z = \frac{(1+3i)(2+i)}{(z-1)(z+1)} = -\frac{1+7i}{5} \) so that \( \text{Re}(z) = -1/5, \text{Im}(z) = 7/5 \) and \( |z| = \sqrt{2} \). (c) \( z = -1 + i\sqrt{3} \) so that \( \text{Re}(z) = -1, \text{Im}(z) = \sqrt{3} \) and \( |z| = \sqrt{2} \).

\[ \text{[a=1, b=2, c=0]} \]

3. (a) \( \sqrt{3} - i = 2 \text{cis}\left(\frac{-\pi}{6}\right) \), \( (b) \frac{3}{1+2i} = \frac{3-6i}{5} = 3/\sqrt{5} \text{cis}(\text{arctan}(-2)) \), \( (c) \frac{w}{w} = (1/5)(-3 - 4i) = \text{cis}\left(\frac{-\pi + \text{arctan}(4/3)}{2}\right) \)

Note: I am happy to take approximations from a calculator in the Class Exercises but there are no calculators allowed in the final exam.

\[ \text{[a=2, b=2, c=0]} \]

4. If \( 2z = z^3 \) then \( 2|z| = |z|^3 \) so one solution is \( z = 0 \). If \( z \neq 0 \) then \( |z| = \sqrt{2} \) so that \( z = \sqrt{2} \text{cis}(\theta) \). Hence \( \text{cis}(\theta - \theta) = \text{cis} 3\theta \) or \( \text{cis} 4\theta = 1 \). Thus \( \theta \) is \( 0, \pi/2, \pi \) or \( 3\pi/2 \). So there are five possible solutions where \( z \) equals: \( 0, \sqrt{2}, i\sqrt{2}, -\sqrt{2}, \) or \( -i\sqrt{2} \). Its also possible to solve this by letting \( z = x + iy \) and expanding.

\[ [3] \]

5. (a) \( z = \text{cis}(\pi/6) \) so that \( z^2 = \text{cis}(2\pi/6) = 1/2 + i\sqrt{3}/2 \) \( (b) z^{100} = \text{cis}(100\pi/6) = \text{cis}(16\pi/3 + 2\pi/3) = \text{cis}(2\pi/3) = -1/2 + i\sqrt{3}/2 \).

\[ \text{[a=1, b=2]} \]

6. (a) \( i = \text{cis}(\pi/2) \) so that the square roots of \( i \) are \( \text{cis}(\pi/4) = 1/\sqrt{2} + i/\sqrt{2} \) and \( \text{cis}(\pi/4 + \pi) = \text{cis}(5\pi/4) = -1/\sqrt{2} - i/\sqrt{2} \). (b) The cube roots are \( \text{cis}(\pi/6) = \sqrt{3}/2 + i/2, \text{cis}(\pi/6 + 2\pi/3) \) \( = \text{cis}(5\pi/6) = -\sqrt{3}/2 + i/2 \) and \( \text{cis}(\pi/6 + 4\pi/3) = \text{cis}(9\pi/6) = \text{cis}(3\pi/2) = -i \).

\[ \text{[a=0, b=2]} \]

7. (a) Circle of radius 3 around the point \( 1 + i \). (b) \( \text{Re}(z + i) = 3 \) is the same as \( \text{Re}(z) = 3 \) which is a vertical line through 3. (c) \( \text{Im}(iz) = \text{Re}(z) \) so this is a vertical line through 2. (d) This is the a disk or radius 2 around the point 3 including the edge of the disk.

\[ \text{[a=2, b=0, c=0, d=0]} \]

8*. 

\[
e^{i\theta} = \exp(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots
\]

\[
= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \ldots
\]

\[
= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + 
\ldots + i(\theta - \frac{\theta^3}{3!} + \ldots)
\]

\[
= \cos(\theta) + i\sin(\theta)
\]

**NOTE:** We cannot afford to make every question. If a question is worth zero marks it means that it has not been marked. Please check your solution against the answers.