

PURE MTH 2005 Real & Complex Analysis II

Class Exercise 2. Handed out Thursday 7th August.

Due at 5.00 pm on Wednesday 20th August 2003

Starred questions will not be marked and are not examinable. They are there as a challenge. Everyone should at least think about them. If you have problems or questions please come and see me or email me at michael.murray@adelaide.edu.au.

1. (a) Let $\epsilon > 0$. Find an N such that whenever $n \geq N$ you must have

$$\left| \frac{5 + 6n}{n + 2} - 6 \right| < \epsilon.$$

(b) Now use (a) and the *formal definition of limit* to show that

$$\lim_{n \rightarrow \infty} \frac{5 + 6n}{n + 2} = 6.$$

2. Find the limits of the following sequences. You do not have to prove them from the formal definition but state carefully any results from class that you apply.

$$(a) \ x_n = \frac{6n^2 + 5n + 2}{(n + 2)(n + 1)} \quad (b) \ x_n = \frac{(-1)^n}{n^2} \quad (c) \ x_n = \frac{\cos(n) + n}{n}$$

3. From the formal definition of Cauchy sequence show that $x_n = \frac{n + 1}{2n + 3}$ is Cauchy. [Hint: Look at the proof in class where we showed that a convergent sequence is Cauchy.]

4. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are Cauchy sequences and we define $c_n = a_n + b_n$ for all $n \in \mathbb{N}$ then $(c_n)_{n=1}^{\infty}$ is a Cauchy sequence.

5. Consider a decimal number $a.a_1a_2a_3\dots$ where $a \in \mathbb{Z}$ and $a_i \in \{0, 1, 2, \dots, 9\}$ for each $i \in \mathbb{N}$. Show that

$$x_n = a + \frac{a_1}{10} + \frac{a_2}{100} + \dots + \frac{a_n}{10^n}$$

is a Cauchy sequence.

6*. Prove the product limit law which says that if $\lim_{n \rightarrow \infty} a_n = A$ and $\lim_{n \rightarrow \infty} b_n = B$ then $\lim_{n \rightarrow \infty} a_n b_n = AB$.

7*. Let $f: [a, b] \rightarrow [a, b]$ be a function with the property that there is some fixed constant $c < 1$ such that $|f(x) - f(y)| < c|x - y|$ for every $x, y \in [a, b]$. (Such a function is sometimes called a *contraction*.) Choose any $x_0 \in [a, b]$ and define a sequence $(x_n)_{n=0}^{\infty}$ by $x_{n+1} = f(x_n)$ for $n \geq 0$.

(a) Show that $|x_{n+1} - x_n| \leq c^n |x_1 - x_0|$ for all $n \in \mathbb{N}$.

(b) Deduce that $|x_{n+k} - x_n| \leq c^n \frac{1-c^k}{1-c} |x_1 - x_0|$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

(c) Hence show that $(x_n)_{n=1}^{\infty}$ is Cauchy.

(d) Since the sequence is Cauchy, it converges. If $L = \lim_{n \rightarrow \infty} x_n$, show that $f(L) = L$ and also show that L is the only real number in $[a, b]$ with this property.