Starred questions will not be marked and are not examinable. They are there as a challenge. Everyone should at least think about them. If you have problems or questions please come and see me or email me at michael.murray@adelaide.edu.au.

1. (a) Let \( \epsilon > 0 \). Find an \( N \) such that whenever \( n \geq N \) you must have
\[
\left| \frac{5 + 6n}{n + 2} - 6 \right| < \epsilon.
\]
(b) Now use (a) and the formal definition of limit to show that
\[
\lim_{n \to \infty} \frac{5 + 6n}{n + 2} = 6.
\]

2. Find the limits of the following sequences. You do not have to prove them from the formal definition but state carefully any results from class that you apply.
(a) \( x_n = \frac{6n^2 + 5n + 2}{(n + 2)(n + 1)} \)
(b) \( x_n = \frac{(-1)^n}{n^2} \)
(c) \( x_n = \frac{\cos(n) + n}{n} \)

3. From the formal definition of Cauchy sequence show that \( x_n = \frac{n + 1}{2n + 3} \) is Cauchy. [Hint: Look at the proof in class where we showed that a convergent sequence is Cauchy.]

4. Show that if \( (a_n)_{n=1}^{\infty} \) and \( (b_n)_{n=1}^{\infty} \) are Cauchy sequences and we define \( c_n = a_n + b_n \) for all \( n \in \mathbb{N} \) then \( (c_n)_{n=1}^{\infty} \) is a Cauchy sequence.

5. Consider a decimal number \( a.a_1a_2a_3 \ldots \) where \( a \in \mathbb{Z} \) and \( a_i \in \{0, 1, 2, \ldots, 9\} \) for each \( i \in \mathbb{N} \). Show that
\[
x_n = a + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_n}{10^n}
\]
is a Cauchy sequence.

6*. Prove the product limit law which says that if \( \lim_{n \to \infty} a_n = A \) and \( \lim_{n \to \infty} b_n = B \) then \( \lim_{n \to \infty} a_nb_n = AB \).

7*. Let \( f: [a, b] \to [a, b] \) be a function with the property that there is some fixed constant \( c < 1 \) such that \( |f(x) - f(y)| < c |x - y| \) for every \( x, y \in [a, b] \). (Such a function is sometimes called a contraction.) Choose any \( x_0 \in [a, b] \) and define a sequence \( (x_n)_{n=0}^{\infty} \) by \( x_{n+1} = f(x_n) \) for \( n \geq 0 \).
(a) Show that \( |x_{n+1} - x_n| \leq c^n |x_1 - x_0| \) for all \( n \in \mathbb{N} \).
(b) Deduce that \( |x_{n+k} - x_n| \leq c^n \frac{1-c^k}{1-c} |x_1 - x_0| \) for all \( n \in \mathbb{N} \) and \( k \in \mathbb{N} \).
(c) Hence show that \( (x_n)_{n=1}^{\infty} \) is Cauchy.
(d) Since the sequence is Cauchy, it converges. If \( L = \lim_{n \to \infty} x_n \), show that \( f(L) = L \) and also show that \( L \) is the only real number in \( [a, b] \) with this property.