1. Answer True or False to each of the following questions. No working is required.
   (a) The real numbers \( \mathbb{R} \) are countable.
   (b) Every non-empty, bounded subset of the real numbers has a least upper bound.
   (c) A bounded sequence of real numbers is convergent.
   (d) A continuous function on \((0, 1)\) is always bounded.
   (e) If \( f \) and \( g \) are continuous functions on \([a, b]\) so is \( f + g \).
   (f) A continuous function is always differentiable.
   (g) A Riemann integrable function is always bounded.
   (h) For any \( \delta > 0 \) there is a partition \( P \) of \([a, b]\) with \( \| P \| < \delta \).
   (i) If \( \sum_{n=1}^{\infty} (x_n + y_n) \) is convergent so also is \( \sum_{n=1}^{\infty} x_n \) and \( \sum_{n=1}^{\infty} y_n \).
   (j) If \( \sum_{n=1}^{\infty} x_n \) is convergent then \( \lim_{n \to \infty} x_n = 0 \).

2. (a) (i) Define what it means for \( I \subseteq \mathbb{R} \) to be an interval.
   (ii) Show that if \( I \) and \( J \) are intervals and \( I \cap J \) contains more than one point then \( I \cap J \) is an interval.
   (b) (i) Define what it means for a set \( X \) to be countable.
   (ii) Prove that \( \mathbb{Z} \), the set of all integers, is countable.

3. Let \( S \subseteq \mathbb{R} \) be a non-empty subset of the real numbers.
   (a) Define what it means for \( S \) to be bounded above.
   (b) If \( S \) is bounded above define the least upper bound \( \sup(S) \).
   (c) State the least upper bound axiom for the real numbers.
   (d) For each of the following sets state if they are bounded above, below or not bounded above or below. Give a bound if there is one. You are not required to prove anything.
      (i) \( (0, \infty) \)
      (ii) \( \{1/n^2 \mid n \in \mathbb{N} = \{1, 2, \ldots\}\} \)
      (iii) \( \{x \mid x^2 - 4 > 0\} \)
      (iv) \( \{-1/n \mid n \in \mathbb{N} = \{1, 2, \ldots\}\} \)

[In addition, candidates are allowed ten minutes before the exam begins, to read the paper.]

Calculators are not permitted.
4. (a) Define what it means for a sequence \( \{x_n\} \) to converge to a number \( L \in \mathbb{R} \).

(b) Find the limit of the following sequences and give a brief justification of your answer.

\[
\text{(i) } \frac{5n + 2n^2 + 6}{(n + 1)(n + 2)} \quad \text{(ii) } \frac{\cos(n)}{n} \quad \text{(iii) } \frac{\sin(n)}{n} \quad \text{(iv) } \frac{\tan(n)}{n}.
\]

(c) Let \( \{x_n\} \) be a sequence of real numbers for which \( \lim_{n \to \infty} x_{2n} = L \) and \( \lim_{n \to \infty} x_{2n-1} = L \). Show that \( \lim_{n \to \infty} x_n = L \).

5. Let \( c \in (a, b) \) and recall that \( (a, b) \setminus \{c\} = (a, c) \cup (c, b) \).

(a) If \( f : (a, b) \setminus \{c\} \to \mathbb{R} \) define what it means for \( f \) to have limit \( L \) as \( x \) approaches \( c \).

(b) Assume that \( f : (a, b) \setminus \{c\} \to \mathbb{R} \) with \( \lim_{x \to c} f(x) = L \) and that \( \{x_n\}_{n=1}^{\infty} \subset (a, b) \setminus \{c\} \) with \( \lim_{n \to \infty} x_n = c \). Show that \( \lim_{n \to \infty} f(x_n) = L \).

(c) Find the limit of the sequence \( \left\{ \sqrt{1 + \frac{1}{n}} \right\}_{n=1}^{\infty} \) and justify your answer.

6. Let \( I \subseteq \mathbb{R} \) be an interval.

(a) Define what it means for \( f : I \to \mathbb{R} \) to be uniformly continuous on \( I \).

(b) Show, using the formal definition, that \( f(x) = 5x - 13 \) is uniformly continuous on \( \mathbb{R} \). Show that \( f + g : I \to \mathbb{R} \) is uniformly continuous. Here \( f + g \) is the function defined by \( (f + g)(x) = f(x) + g(x) \).

7. Let \( c \in (a, b) \).

(a) Define what it means for a function \( f : (a, b) \to \mathbb{R} \) to be differentiable at \( c \) and define the derivative \( f'(c) \) of \( f \) at \( c \).

(b) Show that \( f(x) = x^3 \) is differentiable at any \( c \in \mathbb{R} \) using the formal definition and find the derivative.

(c) Explain carefully why \( f \) defined by

\[
f(x) = \frac{\cos(x + x^2)}{(x^3 + 3)^{1/3}}
\]

is differentiable on all of \( \mathbb{R} \). What is the derivative of \( f \)? (You do not need to prove well known facts about the derivative of the cosine function.)

8. Let \( f : [a, b] \to \mathbb{R} \) be continuous and differentiable on \( (a, b) \).

(a) State Rolle’s Theorem for \( f \).

(b) State the Mean Value Theorem for \( f \) and show how to deduce it from Rolle’s Theorem.

(c) Show that if \( f'(t) = 0 \) for all \( t \in (a, b) \) then \( f \) is constant on \([a, b]\).

9. Let \( f : [a, b] \to \mathbb{R} \).

(a) Define the concepts of a partition of \([a, b]\), the norm of a partition and a tagged partition of \([a, b]\).

(b) If \( ^\ast P \) is a tagged partition of \([a, b]\) define the Riemann sum \( S(f, ^\ast P) \).

(c) Define what it means for \( f \) to be Riemann integrable on \([a, b]\).

10. (a) Define what it means for an infinite series \( \sum_{k=1}^{\infty} a_k \) to converge.

(b) Give an example of an infinite series that does not converge. Briefly justify your answer.

(c) Find the sums of the following series and briefly justify your answers:

\[
\text{(i) } \sum_{k=3}^{\infty} \frac{3^{k-2}}{5^k} \quad \text{(ii) } \sum_{k=1}^{\infty} \frac{1}{k(k+1)}
\]