9.1 Surfaces in 3 dimensions.

In the plane, $f(x, y) = 0$ is usually a curve; in three dimensions $F(x, y, z) = 0$ usually represents a surface. Often we can solve for z and write $z = f(x, y)$, where f is a function of two variables.

Planes: You know that the linear equation representing a plane is

$$ax + by + cz + d = 0$$

Cylinders: Consider the equation $\frac{x^2}{3^2} + y^2 = 1$ in $\mathbb{R}^3$.

In $\mathbb{R}^2$ this represents an ellipse; as z does not appear, z may take any value whatsoever with x, y being related by $\frac{x^2}{3^2} + y^2 = 1$. We call this an elliptic cylinder in $\mathbb{R}^3$. We can also have parabolic and hyperbolic cylinders.

Quadric surfaces

An important class of surfaces, called the "quadric surfaces", arise from the general quadratic equation in three variables:

$$ax^2 + by^2 + cz^2 + dxy + exz + fyz + gx + hy + iz + j = 0.$$

As in two-dimensions we can write this as a matrix equation.

$$\mathbf{x}^t A\mathbf{x} + K\mathbf{x} + \mathbf{j} = 0$$

where A is a symmetric 3×3 matrix, $K = \begin{bmatrix} g & h & i \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Using exactly the same method as for two dimensions, that is orthogonally diagonalizing A by an orthogonal matrix P (with $\det P = 1$), we can eliminate the xy, xz, yz terms. Completing the square then puts the equation in standard form.

Examples of quadric surfaces

(1) The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(2) The hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

(3) The hyperboloid of two sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(4) The elliptic cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} .$$

(5) The elliptic paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} .$$

(6) The hyperbolic paraboloid

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c} , \quad c > 0.$$