Review

Last lecture we learnt how to orthogonally diagonalise a real symmetric matrix $A$.

Quadratic Forms: Conic Sections

The equation: $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can always be written in matrix form:

$$
\begin{bmatrix}
  x \\
  y 
\end{bmatrix}
\begin{bmatrix}
  a & b/2 \\
  b/2 & c
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
+ 
\begin{bmatrix}
  d & e
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
+ f = 0
$$

or as $x^tAx + Kx + f = 0$

where

$$
A = \begin{bmatrix}
  a & b/2 \\
  b/2 & c
\end{bmatrix}
\quad x = \begin{bmatrix}
  x \\
  y
\end{bmatrix}
\quad \text{and} \quad K = \begin{bmatrix}
  d & e
\end{bmatrix}.
$$

$x^tAx$ is called a quadratic form in two variables. More generally, if $A$ is a symmetric $n \times n$ (real) matrix and $x \in \mathbb{R}^n$, $x^tAx$ is a quadratic form in $n$ variables.
**General procedure for identifying conics.**

**Case 1:** \( K = [0 \ 0] \).

(a) Write equation as \( x^t A x + f = 0 \) where \( A \) is real and symmetric.
(b) Find \( P \) orthogonally diagonalising \( A \) and such that \( \det(P) = 1 \).
(c) Substitute \( x = P x' \) to transform equation to

\[
(x')^t P^t A P x' + f = \lambda_1 (x')^2 + \lambda_2 (y')^2 + f = 0
\]

where \( D \) has diagonal entries \( \lambda_1 \) and \( \lambda_2 \).
(d) Rearrange into standard form to identify conic.

Note: If \( P \) is orthogonal and \( \det(P) = 1 \) then \( P \) is a rotation:

\[
P = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}.
\]

**Case 2:** \( K \neq [0 \ 0] \).

Do (a), (b) and (c) as before to get:

\[
\lambda_1 (x')^2 + \lambda_2 (y')^2 + KPx' + f = 0
\]
or

\[
\lambda_1 (x')^2 + \lambda_2 (y')^2 + c' x' + d' y' + f = 0.
\]
(d) Complete the square to get

\[
\lambda_1 (x' - x_0)^2 + \lambda_2 (y' - y_0)^2 = f'
\]
(e) Substitute \( x'' = x' - x_0 \) and \( y'' = y' = y_0 \) and rearrange into standard form to identify conic.