Review

Last lecture we learnt how to orthogonally diagonalise a real symmetric matrix A.

Quadratic Forms: Conic Sections

The equation: $ax^2 + bxy + cy^2 + dx + ey + f = 0$ can always be written in matrix form:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + f = 0$$

or as $\mathbf{x}^t A\mathbf{x} + K\mathbf{x} + f = 0$

where $\quad A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\quad K = \begin{bmatrix} d & e \end{bmatrix}$.

$\mathbf{x}^t A\mathbf{x}$ is called a quadratic form in two variables. More generally, if A is a symmetric $n \times n$ (real) matrix and $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}^t A\mathbf{x}$ is a quadratic form in n variables.

General procedure for identifying conics.

Case 1: $K = [0\ 0]$.

(a) Write equation as $\mathbf{x}^t A\mathbf{x} + f = 0$ where A is real and symmetric.

(b) Find P orthogonally diagonalising A and such that $\det(P) = 1$.

(c) Substitute $\mathbf{x} = P\mathbf{x}'$ to transform equation to

$$(\mathbf{x}')^t P^t AP\mathbf{x}' + f = (\mathbf{x}')^t D\mathbf{x}' + f = \lambda_1(x')^2 + \lambda_2(y')^2 + f = 0$$

where D has diagonal entries λ_1 and λ_2.

(d) Rearrange into standard form to identify conic.

Note: If P is orthogonal and $\det(P) = 1$ then P is a rotation:

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Case 2: $K \neq [0\ 0]$.

Do (a), (b) and (c) as before to get:

$$\lambda_1(x')^2 + \lambda_2(y')^2 + KP\mathbf{x}' + f = 0$$

or

$$\lambda_1(x')^2 + \lambda_2(y')^2 + c'x' + d'y' + f = 0.$$

(d) Complete the square to get

$$\lambda_1(x' - x_0)^2 + \lambda_2(y' - y_0)^2 = f'$$

(e) Substitute $x'' = x' - x_0$ and $y'' = y' = y_0$ and rearrange into standard form to identify conic.