8.3 Orthogonal Diagonalisation

Definition. An \( n \times n \) matrix \( P \) is said to be orthogonal if it is invertible and \( P^{-1} = P^t \).

Theorem 8.4. The following statements are equivalent for an \( n \times n \) matrix \( P \).

(i) \( P \) is an orthogonal matrix.

(ii) \( P^t P = I \).

(iii) \( PP^t = I \).

(iv) The rows of \( P \) form an orthonormal basis for \( \mathbb{R}^n \).

(v) The columns of \( P \) form an orthonormal basis for \( \mathbb{R}^n \).

Definition. An \( n \times n \) matrix \( A \) is called orthogonally diagonalisable if there exists an orthogonal matrix \( P \) such that \( D = P^{-1}AP = P^t AP \) is diagonal. The matrix \( P \) is said to orthogonally diagonalise the matrix \( A \).

Evidently, an \( n \times n \) matrix \( A \) is orthogonally diagonalisable if and only if it has an orthonormal set of \( n \) real eigenvectors (since \( A \) is diagonalisable and the matrix \( P \) which diagonalises is orthogonal).

Theorem 8.5. A real \( n \times n \) matrix \( A \) is orthogonally diagonalisable if and only if it is symmetric.
The general quadratic equation in two variables

\[ ax^2 + bxy + cy^2 + dx + ey + f = 0 \]

can always be written in matrix form:

\[ x^t Ax + Kx + f = 0 \]

where \( A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \) is a symmetric matrix, \( x = \begin{bmatrix} x \\ y \end{bmatrix} \) and \( K = \begin{bmatrix} d & e \end{bmatrix} \).

The first part of the equation, \( x^t Ax \) is called a quadratic form in two variables.