8.3 Orthogonal Diagonalisation

Definition. An $n \times n$ matrix P is said to be orthogonal if it is invertible and $P^{-1} = P^t$.

Theorem 8.4. The following statements are equivalent for an $n \times n$ matrix P.

(i) P is an orthogonal matrix.

(ii) $P^t P = I$.

(iii) $PP^t = I$.

(iv The rows of P form an orthonormal basis for $\mathbb{R}^n$.

(v) The columns of P form an orthonormal basis for $\mathbb{R}^n$.

Definition. An $n \times n$ matrix A is called orthogonally diagonalisable if there exists an orthogonal matrix P such that $D = P^{-1}AP (= P^t AP)$ is diagonal. The matrix P is said to orthogonally diagonalise the matrix A.

Evidently, an $n \times n$ matrix A is orthogonally diagonalisable if and only if it has an orthonormal set of n real eigenvectors (since A is diagonalisable and the matrix P which diagonalises is orthogonal).

Theorem 8.5. A real $n \times n$ matrix A is orthogonally diagonalisable if and only if it is symmetric.

8.4 Quadratic Forms: Conic Sections

The general quadratic equation in two variables

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

can always be written in matrix form:

$$\underset{\sim}{x}^t A \underset{\sim}{x} + K\underset{\sim}{x} + f = 0$$

where $A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$ is a symmetric matrix, $\underset{\sim}{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $K = \begin{bmatrix} d & e \end{bmatrix}$.

The first part of the equation, $\underset{\sim}{x}^t A \underset{\sim}{x}$ is called a quadratic form in two variables.