Line Bundles. Honours 1996

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1 Introduction

The mathematical motivation for studying vector bundles comes from the example of the tangent bundle TM of a manifold M. Recall that the tangent bundle is the union of all the tangent spaces T_mM for every m in M. As such it is a collection of vector spaces, one for every point of M.

The physical motivation comes from the realisation that the fields in physics may not just be maps $\phi : M \to \mathbb{C}^N$ say, but may take values in *different* vector spaces at each point. Tensors do this for example. The argument for this is partly quantum mechanics because, if ϕ is a wave function on a space-time M say, then what we can know about are expectation values, that is things like:

$$\int_M \langle \phi(x), \phi(x) \rangle dx$$

and to define these all we need to know is that $\phi(x)$ takes its values in a one-dimensional complex vector space with Hermitian inner product. There is no reason for this to be the same one-dimensional Hermitian vector space here as on Alpha Centauri. Functions like ϕ, which are generalisations of complex valued functions, are called *sections* of vector bundles.

We will consider first the simplest theory of vector bundles where the vector space is a one-dimensional complex vector space - line bundles.

1.1 Definition of a line bundle and examples

The simplest example of a line bundle over a manifold M is the *trivial* bundle $\mathbb{C} \times M$. Here the vector space at each point m is $\mathbb{C} \times \{m\}$ which we regard as a copy of $\mathbb{C}$. The general definition uses this as a local model.

Definition 1.1. A complex line bundle over a manifold M is a manifold L and a smooth surjection $\pi\colon L \to M$ such that:

1. Each *fibre* $\pi^{-1}(m) = L_m$ is a a complex one-dimensional vector space.
2. Every $m \in M$ has an open neighbourhood $U \in M$ for which there is a diffeomeorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}$ such that $\varphi(L_m) \subset \{m\} \times \mathbb{C}$ for every m and that moreover the map $\varphi|_{L_m} : L_m \to \{m\} \times \mathbb{C}$ is a linear isomorphism.

Note 1.1. The second condition is called *local triviality* because it says that locally the line bundle looks like $\mathbb{C} \times M$. We leave it as an exercise to show that local triviality makes the map π a submersion (that is it has onto derivative) and the scalar multiplication and vector addition maps smooth. In the quantum mechanical example local triviality means that at least in some local region like the laboratory we can identify the Hermitian vector space where the wave function takes its values with $\mathbb{C}$.

Example 1.1. $\mathbb{C} \times M$ the trivial bundle

Example 1.2. Recall that if $u \in S^2$ then the tangent space at u to S^2 is identified with the set $T_uS^2 = \{v \in \mathbb{R}^3 \mid \langle v, u \rangle = 0\}$. We make this two dimensional real vector space a one dimensional complex vector space by defining $(\alpha + i\beta)v = \alpha.v + \beta.u \times v$. We leave it as an exercise for the reader to show that this does indeed make T_uS^2 into a complex vector space. What needs to be checked is that $[(\alpha + i\beta)\ (\delta + i\gamma)]v = (\alpha + i\beta)\ [(\delta + i\gamma)]v$ and because T_uS^2 is already a real vector space this follows if $i(iv) = -v$. Geometrically this follows from the fact that we have defined multiplication by i to mean rotation through $\pi/2$. We will prove local triviality in a moment.

Example 1.3. If Σ is any surface in $\mathbb{R}^3$ we can use the same construction as in (2). If $x \in \Sigma$ and $\hat{n}_x$ is the unit normal then $T_x\Sigma = \hat{n}_x^\perp$. We make this a complex space by defining $(\alpha + i\beta)v = \alpha v + \beta \hat{n}_x \times v$.

Example 1.4 (Hopf bundle). Define $\mathbb{C}P_1$ to be the set of all lines (through the origin) in $\mathbb{C}^2$. Denote the line through the vector $z = (z^0, z^1)$ by $[z] = [z^0, z^1]$. Note that $[\lambda z^0, \lambda z^1] = [z^0, z^1]$ for any non-zero complex number λ. Define two open sets U_i by

$$U_i = \{[z^0, z^1] \mid z^i \neq 0\}$$

and co-ordinates by $\psi_i \colon U_i \to \mathbb{C}$ by $\psi_0([z]) = z^1/z^0$ and $\psi_1([z]) = z^0/z^1$. As a manifold $\mathbb{C}P_1$ is diffeomorphic to S^2. An explicit diffeomorphism $S^2 \to \mathbb{C}P_1$ is given by $(x, y, z) \mapsto [x + iy, 1 - z]$.

We define a line bundle H over $\mathbb{C}P_1$ by $H \subset \mathbb{C}^2 \times \mathbb{C}P_1$ where

$$H = \{(w, [z]) \mid w = \lambda z \text{ for some } \lambda \in \mathbb{C}^\times\}.$$

We define a projection $\pi \colon H \to \mathbb{C}P_1$ by $\pi((w, [z])) = [z]$. The fibre $H_[z] = \pi^{-1}([z])$ is the set

$$\{(\lambda z, [z]) \mid \lambda \in \mathbb{C}^\times\}$$

which is naturally identified with the line through $[z]$. It thereby inherits a vector space structure given by

$$\alpha(w, [z]) + \beta(w', [z]) = (\alpha w + \beta w', [z]).$$

We shall prove later that this is locally trivial.

1.2 Isomorphism of line bundles

It is useful to say that two line bundles $L \to M$, $J \to M$ are isomorphic if there is a diffeomorphism map $\varphi : L \to J$ such that $\varphi(L_m) \subset J_m$ for every $m \in M$ and such that the induced map $\varphi|_{L_m} : L_m \to J_m$ is a linear isomorphism.

We define a line bundle L to be *trivial* if it is isomorphic to $M \times \mathbb{C}$ the trivial bundle. Any such isomorphism we call a trivialisation of L.

1.3 Sections of line bundles

A section of a line bundle L is like a vector field. That is it is a map $\varphi : M \to L$ such that $\varphi(m) \in L_m$ for all m or more succinctly $\pi \circ \varphi = id_m$.

Example 1.5 (The trivial bundle.). $L = \mathbb{C} \times M$. Every section φ looks like $\varphi(x) = (f(x), x)$ for some function f.

Example 1.6 (The tangent bundle to S^2.). TS^2. Sections are vector fields. Alternatively because each $T_{xS^2} \subset \mathbb{R}^3$ we can think of a section s as a map $s \colon S^2 \to \mathbb{R}^3$ such that $\langle s(x), x\rangle = 0$ for all $x \in S^2$.

Example 1.7 (The Hopf bundle). By definition a section $s \colon \mathbb{C}P_1 \to H$ is a map

$$s \colon \mathbb{C}P_1 \to H \subset \mathbb{C}^2 \times \mathbb{C}P_1$$

which must have the form $[z] \mapsto ([z], w)$. For convenience we will write it as $s([z]) = ([z], s(z))$ where, for any $[z]$ $s \colon \mathbb{C}P_1 \to \mathbb{C}^2$ satisfies $s([z]) = \lambda z$ for some $\lambda \in \mathbb{C}^\times$.

The set of all sections, denoted by $\Gamma(M, L)$, is a vector space under pointwise addition and scalar multiplication. I like to think of a line bundle as looking like Figure 1.

Here O is the set of all zero vectors or the image of the *zero section*. The curve s is the image of a section and thus generalises the graph of a function.

We have the following result:

Proposition 1.1. *A line bundle L is trivial if and only if it has a nowhere vanishing section.*

Proof. Let $\varphi \colon L \to M \times \mathbb{C}$ be the trivialisation then $\varphi^{-1}(m, 1)$ is a nowhere vanishing section.

Conversely if s is a nowhere vanishing section then define a trivialisation $M \times \mathbb{C} \to L$ by $(m, \lambda) \mapsto \lambda s(m)$. This is an isomorphism. □

Note 1.2. . The condition of local triviality in the definition of a line bundle could be replaced by the existence of local nowhere vanishing sections. This shows that TS^2 is locally trivial as it clearly has *local* nowhere-vanishing vector fields. Recall however the so called 'hairy-ball theorem' from topology which tells us that S^2 has no global nowhere vanishing vector fields. Hence TS^2 is not trivial. We shall prove this result a number of times.

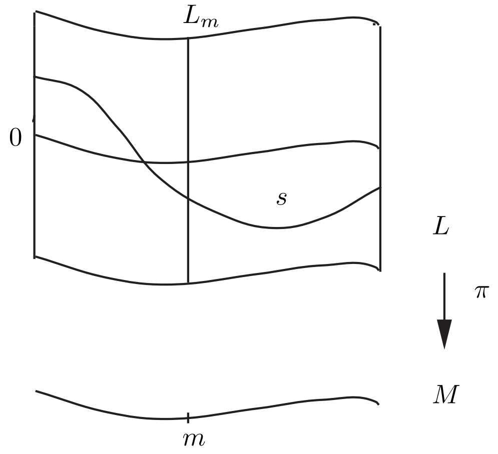

Figure 1: A line bundle.

1.4 Transition functions and the clutching construction

Local triviality means that every property of a line bundle can be understood locally. This is like choosing co-ordinates for a manifold. Given $L \to M$ we cover M with open sets U_α on which there are nowhere vanishing sections s_α. If ξ is a global section of L then it satisfies $\xi|_{U_\alpha} = \xi_\alpha s_\alpha$ for some smooth $\xi_\alpha : U_\alpha \to \mathbb{C}$. The converse is also true. If we can find ξ_α such that $\xi_\alpha s_\alpha = \xi_\beta s_\beta$ for all α, β then they fit together to define a global section ξ with $\xi|_{U_\alpha} = \xi_\alpha s_\alpha$.

It is therefore useful to define $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ by $s_\alpha = g_{\alpha\beta} s_\beta$. Then a collection of functions ξ_α define a global section if on any intersection $U_\alpha \cap U_\beta$ we have $\xi_\beta = g_{\alpha\beta}\xi_\alpha$. The functions $g_{\alpha\beta}$ are called the *transition functions* of L. We shall see in a moment that they determine L completely. It is easy to show, from their definition, that the transition functions satisfy three conditions:

$$
\begin{aligned}
&(1) \quad g_{\alpha\alpha} = 1 \\
&(2) \quad g_{\alpha\beta} = g_{\beta\alpha}^{-1} \\
&(3) \quad g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cap U_\beta \cap U_\gamma
\end{aligned}
$$

The last condition (3) is called the *cocycle condition.*

Proposition 1.2. *Given an open cover $\{U_\alpha\}$ of M and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ satisfying (1) (2) and (3) above we can find a line bundle $L \to M$ with transition functions the $g_{\alpha\beta}$.*

Proof. Consider the disjoint union $\tilde{M}$ of all the $\mathbb{C} \times U_\alpha$. We stick these together using the $g_{\alpha\beta}$. More precisely let I be the indexing set and define $\tilde{M}$ as the subset of $I \times M$ of pairs (α, m) such that $m \in U_\alpha$. Now consider $\mathbb{C} \times \tilde{M}$ whose elements are triples (λ, m, α) and define $(\lambda, m, \alpha) \sim (\mu, n, \beta)$ if $m = n$ and $g_{\alpha\beta}(m)\lambda = \mu$. We leave it as an exercise to show that $\sim$ is an equivalence relation. Indeed ((1) (2) (3) give reflexivity, symmetry and transitivity respectively.)

Denote equivalence classes by square brackets and define L to be the set of equivalence classes. Define addition by $[(\lambda, m, \alpha)] + [(\mu, m, \alpha)] = [(\lambda+\mu, m, \alpha)]$ and scalar multiplication by $z[(\lambda, m, \alpha)] = [(z\lambda, m, \alpha)]$. The projection map is $\pi([(\lambda, m, \alpha)]) = m$. We leave it as an exercise to show that these are all well-defined. Finally define $s_\alpha(m) = [(1, m, \alpha)]$. Then $s_\alpha\ (m) = [(1, m, \alpha)] = [(g_{\alpha\beta}\ (m), m, \beta)] = g_{\alpha\beta}\ (m) s_\beta\ (m)$ as required.

Finally we need to show that L can be made into a differentiable manifold in such a way that it is a line bundle and the s_α are smooth. Denote by W_α the preimage of U_α under the projection map. There is a bijection $\psi_\alpha \colon W_\alpha \to \mathbb{C} \times U_\alpha$ defined by $\psi_\alpha([\alpha, x, z]) = (z, x)$. This is a local trivialisation. If (V, ϕ) is a co-ordinate chart on $U_\alpha \times \mathbb{C}$ then we can define a chart on L by $(\phi_\alpha^{-1}(V), \phi_\alpha \circ \psi_\alpha)$. We leave it as an exercise to check that these charts define an atlas. This depends on the fact that $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathbb{C}^\times$ is smooth. $\square$

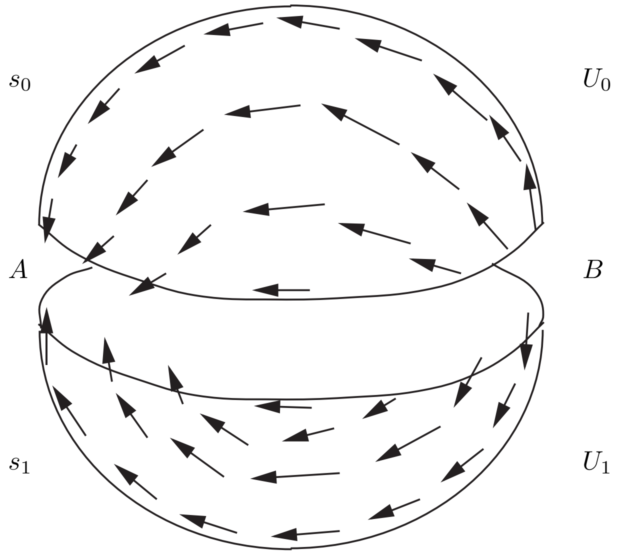

Figure 2: Vector fields on the two sphere.

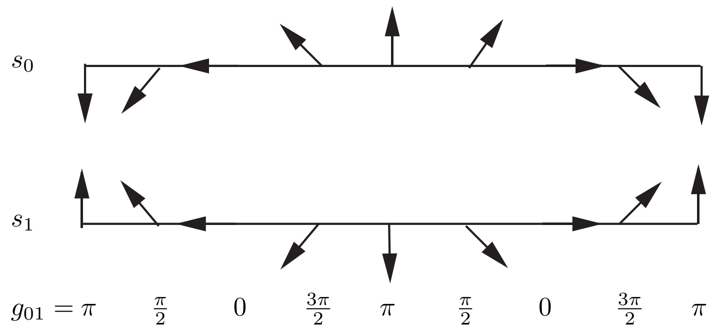

Figure 3: The sections s_0 and s_1 restricted to the equator.

The construction we have used here is called *the clutching construction.* It follows from this proposition that the transition functions capture all the information contained in L. However they are by no means unique. Even if we leave the cover fixed we could replace each s_α by $h_\alpha s_\alpha$ where $h_\alpha : U_\alpha \to \mathbb{C}^\times$ and then $g_{\alpha\beta}$ becomes $h_\alpha g_{\alpha\beta} h_\beta^{-1}$. If we continued to try and understand this ambiguity and the dependence on the cover we would be forced to invent Cêch cohomology and show that that the isomorphism classes of complex line bundles on M are in bijective correspondence with the Cêch cohomology group $H^1(M, \mathbb{C}^\times)$. We refer the interested reader to [11, 8]. We note in passing that the conditions (1), (2) and (3) are equivalent to the usual Cêch cocycle condition that $g_{\beta\gamma} g_{\alpha\beta}^{-1} g_{\alpha\beta} = 1$.

Example 1.8. The tangent bundle to the two-sphere. Cover the two sphere by open sets U_0 and U_1 corresponding to the upper and lower hemispheres but slightly overlapping on the equator. The intersection of U_0 and U_1 looks like an annulus. We can find non-vanishing vector fields s_0 and s_1 as in Figure 2.

If we undo the equator to a straightline and restrict s_0 and s_1 to that we obtain Figure 3.

If we solve the equation $s_0 = g_{01} s_1$ then we are finding out how much we have to rotate s_1 to get s_0 and hence defining the map $g_{01} : U_0 \cap U_1 \to \mathbb{C}^\times$ with values in the unit circle. Inspection of Figure 3 shows that as we go around the equator once s_0 rotates forwards once and s_1 rotates backwards once so that thought of as a point on the unit circle in $\mathbb{C}^\times$ g_{01} rotates around twice. In other words $g_{01} : U_0 \cap U_1 \to \mathbb{C}^\times$ has winding number 2. This two will be important latter.

Example 1.9 (Hopf bundle.). We can define sections $s_i \colon U_\alpha \to H$ by

$$s_0[z] =((1, \frac{z^1}{z^0}), [z]) \tag{1.1}$$

$$s_1[z] =((\frac{z^0}{z^1}, 1), [z]). \tag{1.2}$$

The transition functions are

$$g_{01}([z]) = \frac{z^1}{z^0}.$$

2 Connections, holonomy and curvature

The physical motivation for connections is that you can't do physics if you can't differentiate the fields! So a connection is a rule for differentiating sections of a line bundle. The important thing to remember is that there is no a priori way of doing this - a connection is a *choice* of how to differentiate. Making that choice is something extra, additional structure above and beyond the line bundle itself. The reason for this is that if $L \to M$ is a line bundle and $\gamma : (-\epsilon, \epsilon) \to M$ a path through $\gamma(0) = m$ say and s a section of L then the conventional definition of the rate of change of s in the direction tangent to γ, that is:

$$\lim_{t\to 0} = \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

makes no sense as $s(\gamma(t))$ is in the vector space $L_{\gamma(t)}$ and $s(\gamma(0))$ is in the *different* vector space $L_{\gamma(0)}$ so that we cannot perform the required subtraction.

So being pure mathematicians we make a definition by abstracting the notion of derivative:

Definition 2.1. A connection ∇ is a linear map

$$\nabla : \Gamma(M, L) \to \Gamma(M, T^*M \otimes L)$$

such that for all s in $\Gamma(M, L)$ and $f \in C^\infty(M, L)$ we have the Liebniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

If $X \in T_xM$ we often use the notation $\nabla_X s = (\nabla s)(X)$.

Example 2.1 (The trivial bundle.). $L = \mathbb{C} \times M$. Then identifying sections with functions we see that (ordinary) differentiation d of functions defines a connection. If ∇ is a general connection then we will see in a moment that $\nabla s - ds$ is a 1-form. So *all* the connections on L are of the form $\nabla = d + A$ for A a 1-form on M (*any* 1-form).

Example 2.2 (The tangent bundle to the sphere.). TS^2. If s is a section then $s : S^2 \to \mathbb{R}^3$ such that $s(u) \in T_uS^2$ that is $\langle s(u), u\rangle = 0$. As $s(u) \in \mathbb{R}^3$ we can differentiate it in $\mathbb{R}^3$ but then ds may not take values in T_uS^2 necessarily. We remedy this by defining

$$\nabla(s) = \pi(ds)$$

where π is orthogonal projection from $\mathbb{R}^3$ onto the tangent space to x. That is $\pi(v) = v - \langle x, v\rangle x$.

Example 2.3 (The tangent bundle to a surface.). A surface Σ in $\mathbb{R}^3$. We can do the same orthogonal projection trick as with the previous example.

Example 2.4 (The Hopf bundle.). Because we have $H \subset \mathbb{C}^2 \times \mathbb{C}P_1$ we can apply the same technique as in the previous sections. A section s of H can be identified with a function $s \colon \mathbb{C}P_1 \to \mathbb{C}^2$ such that $s[z] = \lambda z$ for some $\lambda \in \mathbb{C}$. Hence we can differentiate it as a map into $\mathbb{C}^2$. We can then project the result orthogonally using the Hermitian connection on $\mathbb{C}^2$.

The name connection comes from the name infinitesimal connection which was meant to convey the idea that the connection gives an identification of the fibre at a point and the fibre at a nearby 'infinitesimally close' point. Infinitesimally close points are not something we like very much but we shall see in the next section that we can make sense of the 'integrated' version of this idea in as much as a connection, by parallel transport, defines an identification between fibres at endpoints of a path. However this identification is generally path dependent. Before discussing parallel transport we need to consider two technical points.

The first is the question of existence of connections. We have

Proposition 2.1. *Every line bundle has a connection.*

Proof. Let $L \to M$ be the line bundle. Choose an open covering of M by open sets U_α over which there exist nowhere vanishing sections s_α. If ξ is a section of L write it locally as $\xi_{|U_\alpha} = \xi_\alpha s_\alpha$. Choose a partition of unity ρ_α for subordinate to the cover and note that $\rho_\alpha s_\alpha$ extends to a smooth function on all of M. Then define

$$\nabla(\xi) = \sum d\xi_\alpha \rho_\alpha s_\alpha.$$

We leave it as an exercise to check that this defines a connection. □

The second point is that we need to be able to restrict a connection to a open set so that we can work with local trivialisations. We have

Proposition 2.2. *If ∇ is a connection on a line bundle $L \to M$ and $U \subset M$ is an open set then there is a unique connection ∇_U on $L_{|U} \to U$ satisfying*

$$\nabla(s)_{IU} = \nabla_U(s_{|U}).$$

Proof. We first need to show that if s is a section which is zero in a neighbourhood of a point x then $\nabla(s)(x) = 0$. To show this notice that if s is zero on a neighbourhood U of x then we can find a function ρ on M which is 1 outside U and zero in a neighbourhood of x such that $\rho s = s$. Then we have

$$\nabla(s)(x) = \nabla(\rho s)(x) = d\rho(x)s(x) + \rho(x)\nabla(s)(x) = 0.$$

It follows from linearity that if s and t are equal in a neighbourhood of x then $\nabla(s)(x) = \nabla(t)(x)$. If s is a section of L over U and $x \in U$ then we can multiply it by a bump function which is 1 in a neighbourhood of x so that it extends to a section $\hat{s}$ of L over all of M. Then define $\nabla_U(s)(x) = \nabla(\hat{s})(x)$. If we choose a different bump function to extend s to a different section $\tilde{s}$ then $\tilde{s}$ and $\hat{s}$ agree in a neighbourhood of x so that the definition of $\nabla_U(s)(x)$ does not change. □

From now on I will drop the notation $\nabla_{|U}$ and just denote it by ∇.

Let $L \to M$ be a line bundle and $s_\alpha : U_\alpha \to L$ be local nowhere vanishing sections. Define a one-form A_α on U_α by $\nabla s_\alpha = A_\alpha \otimes s_\alpha$. If $\xi \in \Gamma(M, L)$ then $\xi|_{U\alpha} = \xi_\alpha s_\alpha$ where $\xi_\alpha : U_\alpha \to \mathbf{C}$ and

$$\begin{aligned} \nabla\xi|_{U_\alpha} &= d\xi_\alpha s_\alpha + \xi_\alpha \nabla s_\alpha \\ &= (d\xi_\alpha + A_\alpha \xi_\alpha)s_\alpha. \end{aligned} \tag{2.1}$$

Recall that $s_\alpha = g_{\alpha\beta}s_\beta$ so $\nabla s_\alpha = dg_{\alpha\beta}s_\beta + g_{\alpha\beta}\nabla s_\beta$ and hence $A_\alpha s_\alpha = g_{\alpha\beta}^{-1}dg_{\alpha\beta}g_{\alpha\beta}s_\alpha + s_\alpha A_\beta$. Hence

$$A_\alpha = A_\beta + g_{\alpha\beta}^{-1}dg_{\alpha\beta} \tag{2.2}$$

The converse is also true. If $\{A_\alpha\}$ is a collection of 1-forms satisfying the equation (2.2) on $U_\alpha \cap U_\beta$ then there is a connection ∇ such that $\nabla s_\alpha = A_\alpha s_\alpha$. The proof is an exercise using equation (2.1) to define the connection.

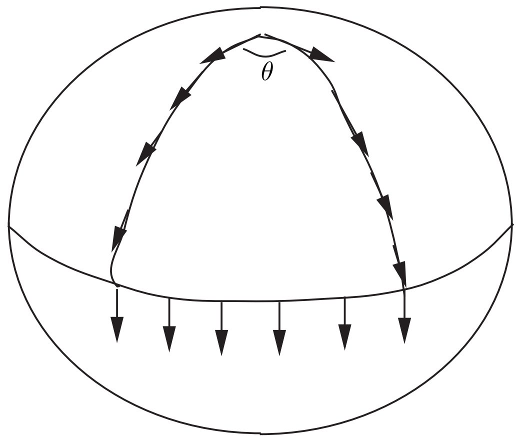

Figure 4: Parallel transport on the two sphere.

2.1 Parallel transport and holonomy

If $\gamma : [0,1] \to M$ is a path and ∇ a connection we can consider the notion of moving a vector in $L_{\gamma(0)}$ to $L_{\gamma(1)}$ without changing it, that is *parallel transporting* a vector from $L_{\gamma(0)}, L_{\gamma(1)}$. Here change is measured relative to ∇ so if $\xi(t) \in L_{\gamma(t)}$ is moving without changing it must satisfy the differential equation:

$$\nabla_{\dot\gamma}\xi = 0 \tag{2.3}$$

where $\dot\gamma$ is the tangent vector field to the curve γ. Assume for the moment that the image of γ is inside an open set U_α over which L has a nowhere vanishing section s_α. Then using (2.3) and letting $\xi(t) = \xi_\alpha(t)s_\alpha(\gamma(t))$ we deduce that

$$\frac{d\xi_\alpha}{dt} = -A_\alpha(\gamma)\xi_\alpha$$

or

$$\xi_\alpha(t) = \exp\bigl(-\int_0^t A_\alpha(\gamma(t))\bigr)\xi_\alpha(0) \tag{2.4}$$

This is an ordinary differential equation so standard existence and uniqueness theorems tell us that parallel transport defines an isomorphism $L_{\gamma(0)} \cong L_{\gamma(t)}$. Moreover if we choose a curve not inside a special open set like U_α we can still cover it by such open sets and deduce that the parallel transport

$$P_\gamma : L_{\gamma(0)} \to L_{\gamma(1)}$$

is an isomorphism. In general P_γ is dependent on γ and ∇. The most notable example is to take γ a *loop* that is $\gamma(0) = \gamma(1)$. Then we define $\mathrm{hol}(\gamma, \nabla)$, the *holonomy* of the connection ∇ along the curve γ by taking any $s \in L_{\gamma(0)}$ and defining

$$P_\gamma(s) = \mathrm{hol}\ (\gamma, \nabla).s$$

Example 2.5. A little thought shows that ∇ on the two sphere preserves lengths and angles, it corresponds to moving a vector so that the rate of change is normal. If we consider the 'loop' in Figure 4 then we have drawn parallel transport of a vector and the holonomy is $\exp(i\theta)$.

2.2 Curvature

If we have a loop γ whose image is in U_α then we can apply (2.4) to obtain

$$\mathrm{hol}\ (\nabla, \gamma) = \exp\ (-\int_\gamma A_\alpha).$$

If γ is the boundary of a disk D then by Stokes' theorem we have

$$\text{hol }(\nabla, \gamma) = \exp - \int_D dA_\alpha. \tag{2.5}$$

Consider the two-forms dA_α. From (2.2) we have

$$\begin{aligned} dA_\alpha &= dA_\beta + d\left(g_{\alpha\beta}^{-1} dg_{\alpha\beta}\right) \\ &= dA_\beta - g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta}^{-1} \wedge dg_{\alpha\beta} + g_{\alpha\beta}^{-1} ddg_{\alpha\beta} \\ &= dA_\beta. \end{aligned}$$

So the two-forms dA_α agree on the intersections of the open sets in the cover and hence define a *global* two form that we denote by F and call the *curvature* of ∇. Then we have

Proposition 2.3. *If $L \to M$ is a line bundle with connection ∇ and Σ is a compact submanifold of M with boundary a loop γ then*

$$hol\ (\nabla, \gamma) = \exp\ - \int_D F$$

Proof. Notice that (2.5) gives the required result if Σ is a disk which is inside one of the U_α. Now consider a general Σ. By compactness we can triangulate Σ in such a way that each of the triangles is in some U_α. Now we can apply (2.5) to each triangle and note that the holonomy up and down the interior edges cancels to give the required result. □

Example 2.6. We calculate the holonomy of the standard connection on the tangent bundle of S^2. Let us use polar co-ordinates: The co-ordinate tangent vectors are:

$$\begin{aligned} \frac{\partial}{\partial\theta} &= (-\sin(\theta)\sin(\phi), \cos(\theta)\sin(\phi), 0) \\ \frac{\partial}{\partial\phi} &= (\cos(\theta)\cos(\phi), \sin(\theta)\cos(\phi), -\sin(\phi)) \end{aligned}$$

Taking the cross product of these and normalising gives the unit normal

$$\begin{aligned} \hat{n} &= (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi)) \\ &= \sin(\phi)\frac{\partial}{\partial\phi} \times \frac{\partial}{\partial\theta} \end{aligned}$$

To calculate the connection we need a non-vanishing section s we take

$$s = (-\sin(\theta), \cos(\theta), 0)$$

and then

$$ds = (-\cos(\theta), -\sin(\theta), 0)d\theta$$

so that

$$\begin{aligned} \nabla s &= \pi(ds) \\ &= ds - < ds, \hat{n} > \hat{n} \\ &= (-\cos(\theta), -\sin(\theta), 0)d\theta \\ &\quad + \sin(\phi)\ (\cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi))d\theta \\ &= (-\cos(\theta)\cos^2(\phi), -\sin(\theta)\cos^2(\phi), \cos(\phi)\sin(\phi))d\theta \\ &= \cos(\phi)\hat{n} \times s \\ &= i\cos(\phi)s \end{aligned}$$

Hence $A = i\cos(\phi)d\theta$ and $F = i\sin(\phi)d\theta \wedge d\phi$. To understand what this two form is note that the volume form on the two-sphere is $\mathrm{vol} = -\sin(\phi)d\theta \wedge d\phi$ and hence $F = i\mathrm{vol}$ The region bounded by the path in Figure 4 has area θ. If we call that region D we conclude that

$$\exp(-\int_D F) = \exp i\theta.$$

Note that this agrees with the previous calculation for the holonomy around this path.

2.3 Curvature as infinitesimal holonomy

The equation $\mathrm{hol}(-\nabla, \partial D) = \exp\left(-\int_D F\right)$ has an infinitesimal counterpart. If X and Y are two tangent vectors and we let D_t be a parallelogram with sides tX and tY then the holonomy around D_t can be expanded in powers of t as

$$\mathrm{hol}\ (\nabla, D_t) = 1 + t^2\ F(X,Y) + 0(t^3).$$

3 Chern classes

In this section we define the Chern class which is a (topological) invariant of a line bundle. Before doing this we collect some facts about the curvature.

Proposition 3.1. *The curvature F of a connection ∇ satisfies:*

(i) $dF = 0$

(ii) If ∇, ∇' are two connections then $\nabla = \nabla' + \eta$ for η a 1-form and $F_\nabla = F_{\nabla'} + d\eta$.

(iii) If Σ is a closed surface then $\frac{1}{2\pi i}\int_\Sigma F_\nabla$ is an integer independent of ∇.

Proof. (i) $dF|_{U\alpha} = d(dA_\alpha) = 0$.
(ii) Locally $A'_\alpha = A_\alpha + \eta\alpha$ as $\eta_\alpha = A'_\alpha - A_\alpha$. But $A_\beta = A_\alpha - g^{-1}_{\alpha\beta}dg_{\alpha\beta}$ and $A'_\beta = A'_\alpha - g^{-1}_{\alpha\beta}dg_{\alpha\beta}$ so that $\eta_\beta = \eta_\alpha$. Hence η is a global 1-form and $F_\nabla = dA_\alpha$ so $F'_\nabla = F_\nabla + d\eta$.
(iii) If Σ is a closed surface then $\partial\Sigma = \emptyset$ so by Stokes' theorem $\int_\Sigma F_\nabla = \int_\Sigma F'_\nabla$. Now choose a family of disks D_t in Σ whose limit as $t \to 0$ is a point. For any t the holonomy of the connection around the boundary of D_t can be calculated by integrating the curvature over D_t or over the complement of D_t in Σ and using Proposition 2.1. Taking into account orientation this gives us

$$\exp(\int_{\Sigma - D_t} F) = \exp(-\int_{D_t} F)$$

and taking the limit as $t \to 0$ gives

$$\exp(\int_\Sigma F) = 1$$

which gives the required result. □

The Chern class, $c(L)$, of a line bundle $L \to \Sigma$ where Σ is a surface is defined to be the integer $\frac{1}{2\pi i}\int_\Sigma F_\nabla$ for any connection ∇.

Example 3.1. For the case of the two sphere previous results showed that $F = -i\mathrm{vol}_{S^2}$. Hence

$$c(TS^2) = \frac{-i}{2\pi i}\int_{S^2} \mathrm{vol} = \frac{-i}{2\pi i}\ 4\pi = -2.$$

Some further insight into the Chern class can be obtained by considering a covering of S^2 by two open sets U_0, U_1 as in Figure 2. Let $L \to S^2$ be given by a transition for $g_{01} : U_0 \cap U_1 \to \mathbf{C}^\times$. Then a connection is a pair of 1-forms A_0, A_1, on U_0, U_1 respectively, such that

$$A_1 = A_0 + dg_{10}g_{10}^{-1} \text{ on } U_0 \cap U_1.$$

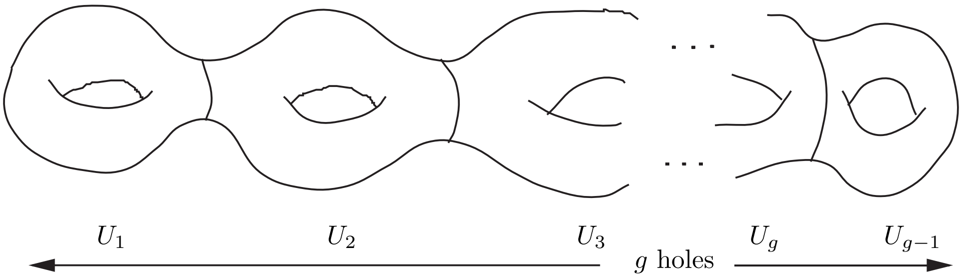

Figure 5: A surface of genus g.

Take $A_0 = 0$ and A_1 to be any extension of $dg_{10}g_{10}^{-1}$ to U_1. Such an extension can be made by shrinking U_0 and U_1 a little and using a cut-off function. Then $F = dA_0 = 0$ on U_0 and $F = dA_1$ on U_1. To find $c(L)$ we note that by Stokes theorem:

$$\int_{S^2} F = \int_{U_1} F = \int_{\partial U_1} A_1 = \int_{\partial U_1} dg_{10}g_{10}^{-1}.$$

But this is just $2\pi i$ the winding number of g_{10}. Hence the Chern class of L is the winding number of g_{10}. Note that we have already seen that for TS^2 the winding number and Chern class are both -2. It is not difficult to go further now and prove that isomorphism classes of line bundles on S^2 are in one to one correspondence with the integers via the Chern class but will not do this here.

Example 3.2. Another example is a surface Σ_g of genus g as in Figure 5. We cover it with g open sets $U_1, \ldots, U_g$ as indicated. Each of these open sets is diffeomorphic to either a torus with a disk removed or a torus with two disks removed. A torus has a non-vanishing vector field on it. If we imagine a rotating bicycle wheel then the inner tube of the tyre (ignoring the valve!) is a torus and the tangent vector field generated by the rotation defines a non-vanishing vector field. Hence the same is true of the open sets in Figure 5. There are corresponding transition functions $g_{12}, g_{23}, \ldots, g_{g-1g}$ and we can define a connection in a manner analogous to the two-sphere case and we find that

$$c(T\Sigma_g) = \sum_{i=1}^{g-1} \text{winding number}(g_{i,i+1}).$$

All the transition functions have winding number -2 so that

$$c(T\Sigma_g) = 2 - 2g.$$

This is a form of the Gauss-Bonnet theorem. It would be a good exercise for the reader familiar with the classical Riemannian geometry of surfaces in $\mathbb{R}^3$ to relate this result to the Gauss-Bonnet theorem. In the classical Gauss-Bonnet theorem we integrate the Gaussian curvature which is the trace of the curvature of the Levi-Civita connection.

So far we have only defined the Chern class for a surface. To define it for manifolds of higher dimension we need to recall the definition of de Rham cohomology [4]. If M is a manifold we have the de Rham complex

$$0 \to \Omega^0(M) \to \Omega^1(M) \to \ldots \to \Omega^m(M) \to 0.$$

where $\Omega^p(M)$ is the space of all p forms on M, the horizontal maps are d the exterior derivative and $m = \dim(M)$. Then $d^2 = 0$ and it makes sense to define:

$$H^p(M) = \frac{\text{kernel } d : \Omega^p\ (M) \to \Omega^{p+1}\ (M)}{\text{image } d : \Omega^{p-1}\ (M) \to \Omega^p\ (M)}$$

This is the pth de Rham cohomology group of M - a finite dimensional vector space if M is compact or otherwise well behaved.

The general definition of $c(L)$ is to take the cohomology class in $H^2(M)$ containing $\frac{1}{2\pi i}F_\nabla$ for some connection.

It is a standard result [4] that if M is oriented, compact, connected and two dimensional integrating representatives of degree two cohomology classes defines an isomorphism

$$\begin{array}{rcl} H^2(M) & \to & \mathbb{R} \\ [\omega] & \mapsto & \int_M \omega \end{array}$$

where $[\omega]$ is a cohomology class with representative form ω. Hence we recover the definition for surfaces.

4 Vector bundles and gauge theories

Line bundles occur in physics in electromagnetism. The electro-magnetic tensor can be interpreted as the curvature form of a line bundle. A very nice account of this and related material is given by Bott in [3]. More interesting however are so-called non-abelian gauge theories which involve vector bundles.

To generalize the previous sections to a vector bundles E one needs to work through replacing $\mathbb{C}$ by $\mathbb{C}^n$ and $\mathbb{C}^\times$ by $GL(n, \mathbb{C})$. Now non-vanishing sections and local trivialisations are not the same thing. A local trivialisation corresponds to a *local frame*, that is n local sections $s_1, ..., s_n$ such that $s_1(m), ..., s_n(m)$ are a basis for E_m all m. The transition function is then matrix valued

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{C}).$$

The clutching construction still works.

A connection is defined the same way but locally corresponds to matrix valued one-forms A_α. That is

$$\nabla|_{U\alpha}(\Sigma_i \xi^i s_i) = \Sigma_i (d\xi i + \Sigma_j A^i_{\alpha j} \xi^j) s_i$$

and the relationship between A_β and A_α is

$$A_\beta = g^{-1}_{\alpha\beta} \; A_\alpha \; g_{\alpha\beta} + g^{-1}_{\alpha\beta} \; dg_{\alpha\beta}.$$

The correct definition of curvature is

$$F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$$

where the wedge product involves matrix multiplication as well as wedging of one forms. We find that

$$F_\beta = g^{-1}_{\alpha\beta} \; F_\alpha \; g_{\alpha\beta}$$

and that F is properly thought of as a two-form with values in the linear operators on E. That is if X and Y are vectors in the tangent space to M at m then $F(X, Y)$ is a linear map from E_m to itself.

We have no time here to even begin to explore the rich geometrical theory that has been built out of gauge theories and instead refer the reader to some references [1, 2, 6, 7].

We conclude with some remarks about the relationship of the theory we have developed here and classical Riemannian differential geometry. This is of course where all this theory began not where it ends! There is no reason in the above discussion to work with complex vector spaces, real vector spaces would do just as well. In that case we can consider the classical example of tangent bundle TM of a Riemannian manifold. For that situation there is a special connection, the Levi-Civita connection. If $(x^1, \ldots, x^n)$ are local co-ordinates on the manifold then the Levi-Civita connection is often written in terms of the Christoffel symbols as

$$\nabla_{\frac{\partial}{\partial x^i}}(\frac{\partial}{\partial x^j}) = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

The connection one-forms are supposed to be matrix valued and they are

$$\sum_i \Gamma^k_{ij} dx^i.$$

The curvature F is the Riemann curvature tensor R. As a two-form with values in matrices it is

$$\sum_{ij} R^k_{ijk} dx^i \wedge dx^j.$$

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