Lie Algebras IV 2008

Root system of $sl(n, \mathbb{C})$

As usual let E_{ij} be the matrix with a 1 in the (i, j) the position and 0's everywhere else. Let $D_{ij} = E_{ii} - E_{jj}$. Let $h_i = D_{ii+1}$. A basis for $sl(n, \mathbb{C})$ is given by the $h_1, \dots, h_{n-1}$ and the E_{ij} for $i \neq j$. Let H be the subalgebra of diagonal (traceless) matrices which we have seen in lectures is a Cartan subalgebra. Let $\epsilon_j \colon H \to \mathbb{C}$ be defined by $\epsilon_j(\sum_{i=1}^n x_i E_{ii}) = x_j$. We have already seen that the roots are given by

$$\Phi = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}$$

and the $\epsilon_i - \epsilon_j$ root space is spanned by E_{ij}. Note that this makes sense as H has dimension $n - 1$ and $sl(n, \mathbb{C})$ has dimension $n^2 - 1$ so that we expect that there are $n^2 - 1 - (n - 1) = n^2 - 1$ roots. Note also that the ϵ_i are not a basis of H^* as there are too many of them and $\epsilon_1 + \cdots + \epsilon_n = 0$. I leave it as an exercise to show that $\epsilon_1 - \epsilon_2, \dots, \epsilon_{n-1} - \epsilon_n$ is a basis of H^*.

If $\alpha = \epsilon_i - \epsilon_j$ then the standard basis of $sl(\alpha)$ is given by $h_\alpha = D_{ij}$, $e_\alpha = E_{ij}$ and $f_\alpha = E_{ji}$. This is easy to check as

$$\begin{aligned}
[h_\alpha, e_\alpha] &= [D_{ij}, E_{ij}] \\
&= [E_{ii} - E_{jj}, E_{ij}] \\
&= [E_{ii}, E_{ij}] - [E_{jj}, E_{ij}] \\
&= E_{ij} - (-1)E_{ij} \\
&= 2E_{ij}
\end{aligned}$$

Likewise $[h_\alpha, f_\alpha] = [D_{ij}, E_{ji}] = -[D_{ji}, E_{ji}] = -2E_{ji}$ and $[e_\alpha, f_\alpha] = [E_{ij}, E_{ji}] = E_{ii} - E_{jj} = D_{ij}$.

There is a theorem which we didn't prove that tells us the Killing form is a multiple of the form $\mathrm{tr}(XY)$ but we can compute directly when we just want the Killing form on H. It suffices to compute $\kappa(h_i, h_j)$. As $sl(n, \mathbb{C})$ is an ideal in $gl(n, \mathbb{C})$ s we can compute the Killing form in the latter space which makes things slightly easier. In $gl(n, \mathbb{C})$

$$\kappa(h_i, h_j) = \kappa(E_{ii}, E_{jj}) - \kappa(E_{ii}, E_{j+1j+1}) - \kappa(E_{i+1i+1}, E_{jj}) + \kappa(E_{i+1i+1} E_{j+1j+1}).$$

so it suffices to compute $\mathrm{tr}(\mathrm{ad}(E_{ii})\,\mathrm{ad}(E_{jj}))$ in $gl(n, \mathbb{C})$. We have

$$\mathrm{ad}(E_{ii})\,\mathrm{ad}(E_{jj})(E_{kl}) = (\delta_{ik} - \delta_{li})(\delta_{jk} - \delta_{lj})E_{kl}$$

so that

$$\begin{aligned}
\mathrm{tr}(\mathrm{ad}(E_{ii})\,\mathrm{ad}(E_{jj})) &= \sum_{kl} (\delta_{ik} - \delta_{li})(\delta_{jk} - \delta_{lj}) \\
&= \sum_{kl} (\delta_{ik}\delta_{jk} - \delta_{ik}\delta_{jl} - \delta_{li}\delta_{jk} + \delta_{li}\delta_{lj} \\
&= n\delta_{ij} - 1 - 1 + n\delta_{ij} \\
&= 2n(\delta_{ij} - 1).
\end{aligned}$$

But we also have $\mathrm{tr}(E_{ii}E_{jj}) = \delta_{ij}$ so that $\mathrm{tr}(\mathrm{ad}(E_{ii})\,\mathrm{ad}(E_{jj})) = 2n\,\mathrm{tr}(E_{ii}E_{jj})$ and it is easy to then show that

$$\kappa(h_i, h_j) = 2n\,\mathrm{tr}(h_i h_j).$$

As we saw in lectures a root system is unchanged if we scale the inner product. So we may as well work with $(1/2n)\kappa$ which is more convenient. It is easy to identify H with a subspace of $\mathbb{C}^n$ by mapping any diagonal matrix to the vector of its diagonal entries. With this choice the roots are the vectors $\alpha_{ij} = e_i - e_j$ which we think of as living in $\mathbb{R}^n$ and the inner product of two roots is just the usual inner product on $\mathbb{R}^n$ restricted to the $n - 1$ dimensional subspace

$$E = \{x \in \mathbb{R}^n \mid x^1 + x^2 + \cdots + x^n = 0\}.$$

The roots are

$$\Phi = \{e_i - e_j \mid i \neq j\}$$

if, as usual, e_i is the vector with a one in the ith place and zeros elsewhere. If we want to find a base for Φ then we note that $(e_i - e_j)(z) = z_i - z_j$ so we need to choose a $z \in E$ which has all its components distinct.

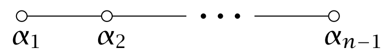
One choice is $z_1 > z_2 > \cdots > z_n$ and then the base of simple roots is given by $\alpha_1 = e_1 - e_2, \ldots, \alpha_{n-1} = e_{n-1} - e_n$. We have

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } |i-j| = 0 \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

so the Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

and the Dynkin diagram is



Finally consider $sl(3, \mathbb{C})$ where we can draw the root system. We take as a base of simple roots the roots $\alpha_1 = e_1 - e_2 = (1, -1, 0)$ and $\alpha_2 = e_2 - e_3 = (0, 1, -1)$. We have $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 2$ and $(\alpha_1, \alpha_2) = -1$. Hence the angle between α_1 and α_2 is $2\pi/3$ and we have the root system A_2: