Lie Algebras IV 2009

Examples of $sl(2,\mathbb{C}$ representations.

We know from lectures that with respect to the basis $X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d$ the matrices of $\psi(e)$, $\psi(f)$ and $\psi(h)$ are given by

$$\psi(e) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \psi(f) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d & 0 & \cdots & 0 & 0 \\ 0 & d-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and

$$\psi(h) = \begin{bmatrix} d & 0 & \cdots & 0 & 0 \\ 0 & d-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d+2 & 0 \\ 0 & 0 & \cdots & 0 & -d \end{bmatrix}$$

We consider the low dimensional cases. $d = 0$ is just the trivial representation where all matrices are 1×1 matrices with a single 0 in them. If $d = 1$ we obtain

$$\psi(e) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \psi(f) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and we see that this is the natural or defining representation of $sl(2,\mathbb{C})$ on $\mathbb{C}^2$.

If $d = 2$ we obtain

$$\psi(e) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \psi(f) = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \psi(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Because this is three-dimensional and the adjoint representation is three-dimensional and irreducible we know that this must be the adjoint representation. But to get the matrices above we have to choose the right basis of $sl(2,\mathbb{C})$ in which to expand the matrices.

Recall that $sl(2,\mathbb{C})$ has commutation relations: $[e,f] = h$, $[h,e] = 2e$ and $[h,f] = -2f$. If we take the basis of $sl(2,\mathbb{C})$ a s $\{e,h,f\}$ then we need to write $[e,e] = 0$, $[e,h] = -2e$ and $[e,f] = h$ and likewise for the h and f action to get

$$\psi(e) = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \psi(f) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad \psi(h) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

which is not quite right. To get the correct basis we can start guessing but it is better to apply the proof of the Theorem in class. This told us to send $e \mapsto X^2$, $[f,e] = -h \mapsto f(X^2) = 2XY$ and $[f,[f,e]] = [f,-h] = -2f \mapsto f^2(X^2) = 2Y^2$. Or if we use the inverse of this isomorphism we map $X^2 \mapsto e$, $XY \mapsto -h/2$ and $Y^2 \mapsto -f$. So take the basis $v_1 = e$, $v_2 = -h/2$ and $v_3 = -f$ for $sl(2,\mathbb{C})$. Then we obtain:

$$[e,v_1] = 0, [e,v_2] = v_1, [e,v_3] = 2v_2,$$

which gives the matrix above for $\psi(e)$. Similarly

$$[f,v_1] = 2v_2, [f,v_2] = v_3, [f,v_3] = 0$$

which is the matrix above for $\psi(f)$ and finally

$$[h,v_1] = 2v_1, [h,v_2] = 0, [h,v_3] = -2v_3$$

which is the matrix for $\psi(f)$. Hence V_2 is the adjoint representation.