PLAN FOR LIE ALGEBRAS IV 2009

Lecture 1.

1. Introduction to Lie algebras

Handout: Information about the course.

Discussion of what Lie algebras are about. Campbell Baker Hausdorff formula.

Note 1.1. Throughout we will use the notation $\mathbb{F}$ to denote either of $\mathbb{C}$ or $\mathbb{R}$. It is actually possible to define and discuss Lie algebras over any field but we will not be doing that.

Definition 1.1. Let V and V' be $\mathbb{F}$ vector spaces. A function $m\colon V \times V \to V'$ is called *bilinear* if for all $u, v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in \mathbb{F}$ we have

$$m(\alpha_1 v_1 + \alpha_2 v_2, u) = \alpha_1 m(v_1, u) + \alpha_2 m(v_2, u) \qquad \text{and,}$$
$$m(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 m(u, v_1) + \alpha_2 m(u, v_2).$$

Definition 1.2. An $\mathbb{F}$ vector space A is called an *algebra* if it has a bilinear map

$$A \times A \to A$$
$$(a, b) \mapsto ab$$

usually called product or multiplication.

Example 1.1. $\mathbb{F}$

Example 1.2. The space of all infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{C}$, $C^\infty(\mathbb{R}, \mathbb{C})$, with pointwise multiplication of functions.

Example 1.3. The space $M_n(\mathbb{C})$ of all $n \times n$ complex matrices with matrix multiplication.

Definition 1.3. A *Lie algebra* L is an algebra with a product

$$L \times L \to L$$
$$(x, y) \mapsto [x, y]$$

satisfying

$$[x, x] = 0$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in L$.

Note 1.2. We call $[x, y]$ the (Lie) bracket of x and y. The first of these conditions is called anti-symmetry and the second is known as the Jacobi identity.

Note 1.3. We can use anti-symmetry to show that $[x, y] = -[y, x]$ for all $x, y \in L$.

Example 1.4. L any vector space and $[x, y] = 0$ for all $x, y \in L$. This is a Lie algebra.

Definition 1.4. A Lie algebra is called *abelian* if $[x, y] = 0$ for all $x, y \in L$.

Example 1.5. $M_n(\mathbb{C})$ is a Lie algebra with the Lie bracket the commutator of matrices: $[X, Y] = XY - YX$.

Example 1.6. Let V be any vector space and $gl(V)$ be the space of all linear maps $f\colon V \to V$ and define $[f, g] = f \circ g - g \circ f$ where $f, g \in gl(V)$. This is a Lie algebra called the *general linear algebra* of V.

Note 1.4. Sometimes we also write $M_n(\mathbb{C}) = gl(n, \mathbb{C}) = gl_n(\mathbb{C})$.

Example 1.7. For $x, y \in \mathbb{R}^3$ let $[x, y] = x \times y$ the vector or cross product. This is a Lie algebra.

Definition 1.5. Let A be an algebra. Call a linear map $d\colon A \to A$ a *derivation* if for all $a, b \in A$ we have $d(ab) = d(a)b + ad(b)$. Denote by $\mathrm{Der}(A)$ the set of all derivations.

Note 1.5. $\mathrm{Der}(A) \subseteq gl(A)$ is a vector subspace.

Definition 1.6. If A is an algebra and B is a vector subspace of A with $b_1 b_2 \in B$ for all $b_1, b_2 \in B$ then we call B a *subalgebra* of A.

Definition 1.7. If L is a Lie algebra and J is a vector subspace with $[x, y] \in J$ for all $x, y \in J$ we call J a (Lie) *subalgebra* of L.

Proposition 1.8. *If A is an algebra then $\mathrm{Der}(A)$ is a Lie subalgebra of $gl(A)$.*

Example 1.8. Let $A = C^\infty(\mathbb{R}^3, \mathbb{R})$ be the space of all infinitely differentiable functions on $\mathbb{R}^3$. Let $X = (X^1, X^2, X^3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field on $\mathbb{R}^3$ where each X^i is also infinitely differentiable. If $f \in A$ define

$$X(f) = \sum_{i=1}^{3} X_i \frac{\partial f}{\partial x^i}.$$

Then X thought of as a function $X\colon A \to A$ is a derivation. Moreover $[X, Y]$ has components

$$[X, Y]_i = \sum_{j=1}^{3} \left(X_j \frac{\partial Y_i}{\partial x^j} - Y_j \frac{\partial X_i}{\partial x^j} \right).$$

Lecture 2.

Definition 1.9. Let L be a Lie algebra. A linear map $d\colon L \to L$ is called a *derivation* if $d([x, y]) = [d(x), y] + [x, d(y)]$ for all $x, y \in L$.

Lemma 1.10. *Let L be a Lie algebra and $x \in L$. The map $\mathrm{ad}_x\colon L \to L$ defined by $\mathrm{ad}_x(y) = [x, y]$ is a derivation and the map $\mathrm{ad}\colon L \to \mathrm{Der}(L)$ defined by $\mathrm{ad}(x) = \mathrm{ad}_x$ is a Lie algebra homomorphism.*

Note 1.6. The map $\mathrm{ad}\colon L \to \mathrm{Der}(L)$ is called the adjoint map or the adjoint representation of L.

Example 1.9. Let $sl_n(\mathbb{C}) \subseteq gl_n(\mathbb{C})$ be the set of all matrices with zero trace. This is a Lie subalgebra of $gl_n(\mathbb{C})$ called the *special linear algebra*.

Example 1.10. The subspaces $b_n(\mathbb{C}) = b(n, \mathbb{C})$ and $n_n(\mathbb{C}) = n(n, \mathbb{C})$ of upper triangular and strictly upper triangular matrices are Lie subalgebras of $gl_n(\mathbb{C})$.

1.1. **Homorphisms.**

Definition 1.11. If L and J are Lie algebras a linear map $\phi\colon L \to J$ is a called a *Lie algebra homomorphism* if $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in L$. If ϕ is a bijection we call it a *Lie algebra isomorphism* and write $L \simeq J$.

Exercise 1.1. The adjoint map is a homomorphism of Lie algebras.

Definition 1.12. If I is a subspace of a Lie algebra L we say that I is an *ideal* if $[x, y] \in I$ for all $x \in L$ and $y \in I$.

Lemma 1.13. *The kernel of a homomorphism of Lie algebras is an ideal and the image is a subalgebra.*

Example 1.11. The kernel of the adjoint map of L is the *centre* of L,

$$Z(L) = \{x \in L \mid [x, y] = 0 \forall y \in L\}$$

Example 1.12. The image of the adjoint map is the subalgebra of *inner derivations* $\mathrm{IDer}(L) \subseteq \mathrm{Der}(L)$.

Definition 1.14. If A and B are subspaces of a Lie algebra L we define

$$[A, B] = \mathrm{span}\{[a, b] \mid a \in A, b \in B\}.$$

and

$$A + B = \{a + b \mid a \in A, b \in B\}$$

Note 1.7. If $z \in [A, B]$ then there exist $x_1, \ldots, x_r \in A$ and $y_1, \ldots, y_r \in B$ such that

$$z = \sum_{i=1}^{r} [x_i, y_i]$$

Proposition 1.15. *If A and B are ideals in a Lie algebra L then so also are $[A, B]$, $A + B$ and $A \cap B$.*

Example 1.13. If L is a Lie algebra then $L' = [L, L]$ is an ideal called the *derived algebra* of L.

Lecture 3.

Proposition 1.16. *The subalgebra of inner derivations* $\mathrm{IDer}(L)$ *is an ideal in* $\mathrm{Der}(L)$.

Proposition 1.17. *If $I \subseteq L$ is an ideal in a Lie algebra consider the quotient vector space L/I and define a bracket operation by $[x + I, y + I] = [x, y] + I$. This is well-defined and makes L/I a Lie algebra and the linear surjective map $L \to L/I$ which sends x to $x + I$ is a Lie algebra homomorphism.*

Note 1.8. We call the map $L \to L/I$ the (canonical) projection.

Proposition 1.18. *A subset $I \subseteq L$ is an ideal if and only if it is the kernel of a Lie algebra homomorphism.*

Theorem 1.19 (Isomorphism theorems).

a) Let $\psi\colon L_1 \to L_2$ be a homomorphism between Lie algebras. Then $\ker(\phi) \subseteq L_1$ *is an ideal,* $\mathrm{im}(\phi) \subseteq L_2$ *a subalgebra and there is an isomorphism*

$$\begin{aligned} \bar{\phi}\colon L_1/\ker(\psi) &\to \mathrm{im}(\psi) \\ x + \ker(\psi) &\mapsto \psi(x) \end{aligned}$$

b) If I and J are ideals in a Lie algebra L then $(I + J)/I \cong I/I \cap J$.
c) If $I \subseteq J$ are ideals of L then J/I is an ideal of L/I and $(L/I)/(J/I) \cong L/J$.

1.2. **Direct sums.**

Proposition 1.20. *If L_1 and L_2 are Lie algebras let $L = L_1 \oplus L_2$ and define a map $L \times L \to L$ by letting*

$$[(x_1, x_2), (y_1, y_2)] \mapsto ([x_1, y_1], [x_2, y_2])$$

for all $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$. This defines a Lie bracket making $L = L_1 \oplus L_2$ into a Lie algebra which we call the direct sum *of L_1 and L_2.*

Proposition 1.21. *Suppose a Lie algebra L has ideals I_1 and I_2 such that $L = I_1 + I_2$ and $I_1 \cap I_2 = 0$ then the map*

$$\begin{aligned} I_1 \oplus I_2 &\to L \\ (x_1, x_2) &\mapsto x_1 + x_2 \end{aligned}$$

is an isomorphism of Lie algebras.

Note 1.9. If I and J are ideals in L then $[I, J] \subseteq I \cap J$.

Lecture 4.

1.3. **Ideals and homomorphisms.**

Lemma 1.22. *Let $\phi\colon L \to J$ be an homomorphism of Lie algebras. Then*

(1) If $I \subseteq J$ is an ideal then $\phi^{-1}(I) \subseteq L$ is an ideal.
(2) If ϕ is surjective and $I \subseteq L$ is an ideal then $\phi(I) \subseteq J$ is an ideal

Proposition 1.23. *Let $I \subseteq L$ be an ideal and consider $\phi\colon L \to L/I$. Then the following is a bijection:*

$$\begin{aligned} \{J \mid J \text{ is an ideal in } L \text{ and } I \subseteq J\} &\to \{K \mid K \text{ is an ideal in } L/I\} \\ J &\mapsto J/I \end{aligned}$$

2. Low-dimensional Lie algebras

We classify all complex Lie algebras L with $\dim(L) \leq 3$. Note that in each dimension there is a unique abelian Lie algebra. Also if L is not abelian then $L\,L' \neq 0$ and $Z(L) \neq L$.

Case 1: $\dim(L) = 1$. If $\dim(L) = 1$ then L is the unique one-dimensional abelian Lie algebra.

Case 2: $\dim(L) = 2$. If L is not abelian there is a two-dimensional Lie algebra with basis $\{x, y\}$ and Lie bracket determined by $[x, y] = x$.

Theorem 2.1. *If L is a two-dimensional, non-abelian, complex Lie algebra then L is isomorphic to the two-dimensional Lie algebra described above.*

Case 3: $\dim(L) = 3$, $L' \subseteq Z(L)$ and $\dim(L') = 1$.

Theorem 2.2. *Up to isomorphism the Lie algebra $\mathfrak{n}(3, \mathbb{C})$ of all strictly upper-triangular, three by three matrices is the unique three-dimensional Lie algebra with L' one-dimensional and $L' \subseteq Z(L)$. This is also known as the Heisenberg Lie algebra. It has a basis*

$$f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad z = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

with $[f, g] = z \in Z(L)$.

Example 2.1. Let L_1 be the one-dimensional Lie algebra and L_2 the two-dimensional, non-abelian Lie-algebra and let $L = L_1 \oplus L_2$. Then $L' = 0 \oplus L_2'$ and $Z(L) = L_1 \oplus 0$. Notice that L' is one-dimensional and not contained in $Z(L)$.

Lecture 5.

Theorem 2.3. *The Lie algebra in Example* **??** *is the unique (up to isomorphism) three-dimensional Lie algebra L with L' one-dimensional and L' not contained in $Z(L)$.*

2.1. Review of some linear algebra.

Trace of a linear map Recall that if X and Y are matrices then $\mathrm{tr}(XY) = \mathrm{tr}(YX$ so that $\mathrm{tr}([X, Y]) = 0$ and if G is invertible then $\mathrm{tr}(GXG^{-1}) = \mathrm{tr}(G^{-1}GX) = \mathrm{tr}(X)$. Let $X \colon V \to V$ be a linear map and V a finite-dimensional vector space. If we choose a basis for X we can turn it into a matrix M and define $\mathrm{tr}(X) = \mathrm{tr}(M)$. This is actually independent of the choice of basis because changing the basis will replace M by GMG^{-1} for some invertible matrix G. If $Y \colon V \to V$ then we have $\mathrm{tr}([X, Y]) = 0$.

Jordan canonical form For $a \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ define the $a \times a$ matrix

$$J_a(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}$$

Lecture 6.

Theorem 2.4. *Let $X \colon V \to V$ be a linear map and V a finite-dimensional complex vector space. Then there is a basis of V, $a_1, a_2, \ldots, a_r$ and $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that X has the form of a block diagonal matrix*

$$\begin{bmatrix} J_{a_1}(\lambda_1) & 0_{a_1 a_2} & \cdots & 0_{a_1 a_r} \\ 0_{a_2 a_1} & J_{a_2}(\lambda_2) & \cdots & 0_{a_2 a_r} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{a_r a_1} & 0_{a_r a_2} & \cdots & J_{a_r} \lambda_r \end{bmatrix}$$

where 0_{pq} is a $p \times q$ matrix of zeroes.

Lemma 2.5. *Let L be a three-dimensional complex Lie algebra* $\dim(L') = 2$. *Choose a basis $\{y, z\}$ of L and extend it to a basis of L with x. Then*

(1) L' is abelian
(2) $\mathrm{ad}_x \colon L' \to L'$ is an isomorphism

Proposition 2.6. *Let L be a three-dimensional complex Lie algebra with* $\dim(L') = 2$. *Choose a basis $\{y, z\}$ of L and extend it to a basis of L with a vector x. Assume that* $\mathrm{ad}_x \colon L' \to L'$ *is diagonalizable. Then after rescaling x it has the matrix form*

$$\begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

for $0 \neq \mu \in \mathbb{C}$. Then each choice of μ defines a Lie algebra L_μ satisfying the hypothesis of the proposition. Moreover L_μ is isomorphic to L_ν if and only if $\mu = \nu$ or $\mu = 1/\nu$.

Proposition 2.7. *There is a unique (up to isomorphism) three-dimensional complex Lie algebra L satisfying the following conditions. The commutator subalgebra has dimension* $\dim(L') = 2$ *and if we choose a basis $\{y, z\}$ of L' and extend it to a basis of L with x then* $\mathrm{ad}_x \colon L' \to L'$ *is not diagonalizable.*

Example 2.2. $L = sl(2, \mathbb{C})$ has a basis given by

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with $[e, f] = h$, $[h, e] = 2e$ and $[h, f = -2f$. Therefore the commutator algebra of $sl(2, \mathbb{C})$ is just $sl(2, \mathbb{C})$.

Lemma 2.8. *Let L be a three-dimensional $\mathbb{C}$ Lie algebra such that $L' = L$. Then*

a) If $x \in L$ with $x \neq 0$ then $\mathrm{ad}_x \colon L \to L$ has rank two.
b) $\exists h \in L$ such that $\mathrm{ad}_h \colon L \to L$ has an eigenvector with non-zero eigenvalue.

Proposition 2.9. $sl(2, \mathbb{C})$ *is the unique complex Lie algebra of dimension three which is equal to its commutator subgroup.*

Summary: Non-abelian complex Lie algebras L with $\dim(L) \leq 3$.

$\dim(L) = 1$: None just the abelian Lie algebra.

$\dim(L) = 2$: Just one up to isomorphism. $L_2 = \mathrm{span}\{x, y\}$ with $[x, y] = x$.

$\dim(L) = 3$, $\dim(L') = 1$, $L' \subseteq Z(L)$: Heisenberg Lie algebra

$\dim(L) = 3$, $\dim(L') = 1$, $L' \not\subset Z(L)$: Only the direct sum of the one-dimensional Lie algebra and the non-abelian two-dimensional Lie algebra.

$\dim(L) = 3$, $\dim(L') = 2$: Infinitely many. For each $\mu \in \mathbb{C}$ there is L_μ for with $L_\mu \simeq L_\rho$ if and only if $\mu = \rho$ or $\mu = 1/\rho$.

$\dim(L) = 3$, $\dim(L') = 3$: Only $sl(2, \mathbb{C})$.

Lecture 7.

3. Solvable Lie algebras

Lemma 3.1. *Let I be an ideal of a Lie algebra L. Then L/I is abelian if and only if $L' \subseteq I$.*

Definition 3.2. If L is a Lie algebra the *derived series* is the sequence of ideals $L^{(1)}, L^{(2)}, \ldots$ in L defined by $L^{(1)} = L' = [L, L]$, $L^{(2)} = [L^{(1)}, L^{(1)}$, etc.

Note 3.1. Notice that we have

$$L \supset L^{(1)} \supset L^{(2)} \supset \ldots$$

and $L/[L, L]$ is abelian so $L^{(k)}/L^{(k+1)}$ is abelian.

Definition 3.3. A Lie algebra L is called *solvable* if there is some $m \geq 0$ with $L^{(m)} = 0$.

Example 3.1. Solvable Lie algebras include the Heisenberg Lie algebra, the Lie algebra of upper triangular matrices and any two-dimensional Lie algebra. $sl(2, \mathbb{C})$ is not solvable.

Lemma 3.4. *If L has a collection of ideals*

$$L \supset I_1 \supset I_2 \supset \cdots \supset I_m = 0$$

and I_k/I_{k+1} is abelian for all $1 \leq m-1$ then L is solvable.

Lemma 3.5. *If $\varphi\colon L \to J$ is a surjective homomorphism then $\varphi(L^{(k)}) = J^{(k)}$ for all k.*

Lemma 3.6. *Let L be a Lie algebra. Then*

a) If L is solvable so also is any subalgebra of L or homomorphic image of L.
b) If L has an ideal I with I and L/I solvable then L is solvable.
c) If I and J are solvable ideals of L so also is $I + J$.

3.1. **The radical.**

Corollary 3.7. *If L is finite dimensional it has a unique solvable ideal containing all other solvable ideals.*

Definition 3.8. If L is finite dimensional we call the unique largest solvable ideal the *radical* of L and denote it by $\mathrm{rad}(L)$.

Definition 3.9. A non-zero finite dimensional Lie algebra L is called *semisimple* if $\mathrm{rad}(L) = 0$.

Lecture 8.

Example 3.2. $sl(2, \mathbb{C})$ is semisimple.

Lemma 3.10. *If L is a Lie algebra then $L/\mathrm{rad}(L)$ is semisimple.*

The Plan: We want to classify semisimple Lie algebras. First we will show that every semisimple Lie algebra is the direct sum of simple Lie algebras were:

Definition 3.11. A Lie algebra is simple if it is no ideals other than itself and zero and it is not abelian.

Then the simple Lie algebras are exactly the following: $A_n = sl(n, \mathbb{C})$ for $n \geq 1$, $B_n = so(2n+1, \mathbb{C})$ for $n \geq$, $C_n = sp(n, \mathbb{C})$, for $n \geq 3$, $D_n = so(2n, \mathbb{C})$ for $n \geq 4$ and E_6, E_7, E_8, F_4 and G_2.

Construction of classical Lie algebras. The Lie algebras A_n, B_n, C_n and D_n are called classical Lie algebras and the latter three series are examples of the following construction. Let $S \in gl(n, \mathbb{C})$ and define

$$gl_S(n, \mathbb{C}) = \{x \in gl(n, \mathbb{C}) \mid x^t S = -Sx\}.$$

Then $gl_s(n, \mathbb{C})$ is a Lie algebra. We

$$so(2n, \mathbb{C}) = gl_S(2n, \mathbb{C}) \quad \text{for} \quad S = \begin{bmatrix} 0_{nn} & I_{nn} \\ I_{nn} & 0_{nn} \end{bmatrix} \tag{3.1}$$

$$so(2n+1, \mathbb{C}) = gl_S(2n+1, \mathbb{C}) \quad \text{for} \quad S = \begin{bmatrix} 1 & 0_{1n} & 0_{1n} \\ 0_{n1} & 0_{nn} & I_{nn} \\ 0_{n1} & I_{nn} & 0_{nn} \end{bmatrix} \tag{3.2}$$

$$sp(2n, \mathbb{C}) = gl_S(2n, \mathbb{C}) \quad \text{for} \quad S = \begin{bmatrix} 0_{nn} & I_{nn} \\ -I_{nn} & 0_{nn} \end{bmatrix} \tag{3.3}$$

Lecture 9.

3.2. **Nilpotent Lie algebras.**

Definition 3.12. If L is a Lie algebra we define the *lower central series* L^k by

$$L^1 = L', L^2 = [L, L^1], L^3 = [L, L^2], \ldots.$$

Note 3.2. Each L^k is an ideal and $L \supset L^1 \supset L^2 \supset \ldots$. The name comes from the fact that $L^k/L^{k+1} \subseteq Z(L/L^{k+1})$.

Definition 3.13. A Lie algebra L is called *nilpotent* if $L^k = 0$ for some $k \geq 1$.

Note 3.3. As $L^{(k)} \subseteq L^k$ a nilpotent Lie algebra is solvable. The converse is not true as $b(n, \mathbb{C})$ is solvable but not nilpotent.

Lemma 3.14. *Let L be a Lie algebra. Then*

a) *If L is nilpotent then any subalgebra of L is nilpotent.*
b) *If $L/Z(L)$ is nilpotent then L is nilpotent.*

Note 3.4. Note that we can have I a nilpotent ideal in L and L/I nilpotent but L not nilpotent. For example consider L the non-abelian two-dimensional Lie algebra spanned by $\{x, y\}$ with $[x, y] = x$ and I the span of x.

4. Subalgebras of $gl(V)$

4.1. **Nilpotent maps.**

Definition 4.1. We say that $x \in gl(V)$ is *nilpotent* if there is an m such that $x^m = 0$.

Note 4.1. Let V be a finite-dimensional vector space and $L \subseteq gl(V)$ be a subalgebra. If $x \in L$ note that x^m may not be in L for $m > 1$.

Lemma 4.2. *Let L be a Lie subalgebra of $gl(V)$ and $x \in L$. If x is nilpotent so also is* $\mathrm{ad}(x) \in gl(L)$.

Definition 4.3. A *weight* for a Lie subalgebra $A \subseteq gl(V)$ is a linear map $\lambda \colon A \to \mathbb{F}$ such that

$$V_\lambda = \{v \in V \mid av = \lambda(a)v \,\forall a \in A\} \neq 0.$$

Note 4.2. We call V_λ the λ *weight space*.

Definition 4.4. If $x \in gl(V)$ and W is a subspace of V then we define

$$x(W) = \{x(w) \mid w \in W\}.$$

We say that W is invariant under x if $x(W) \subseteq W$.

Definition 4.5. If $L \subseteq gl(V)$ is a Lie subalgebra and $W \subseteq V$ we call W L-invariant if W is x invariant for all $x \in L$.

Lemma 4.6. *Suppose $A \subseteq L$ is an ideal in a Lie subalgebra L of $gl(V)$. Let*

$$W = \{v \in V \mid av = 0 \forall a \in A\}$$

then W is L-invariant.

$\boxed{\textbf{Lecture 10.}}$

Lemma 4.7. *(Invariance Lemma) Assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let L be a Lie subalgebra of $gl(V)$ and $A \subseteq L$ an ideal. Let $\lambda \colon A \to \mathbb{F}$ be a weight of A. Then the weight space V_λ is L-invariant.*

If $W \subseteq V$ denote by $gl_W(V)$ the subalgebra of all $x \colon V \to V$ such that $x(W) \subseteq W$. For such x there is an induced linear map $\bar{x} \colon V/W \to V/W$ and hence there is a map

$$gl_W(V) \to gl(V/W)$$

which is a homomorphism of algebras and hence Lie algebras.

5. Lie and Engel's Theorems

Theorem 5.1. *(Engel's Theorem) Let V be a vector space and $L \subseteq gl(V)$ a Lie subalgebra such that for all $x \in L$ we have x nilpotent. Then V has a basis in which every $x \in L$ is represented by a strictly upper-triangular matrix.*

Lecture 11.

Proposition 5.2. *Suppose L is a Lie subalgebra of $gl(V)$ such that for all $x \in L$ we have x nilpotent. Then there is a $v \in V$, $v \neq 0$ such that $xv = 0$ for all $x \in L$.*

Theorem 5.3. *(Second version of Engel's theorem) A Lie algebra L is nilpotent if and only if or all $x \in L$ we have* $\mathrm{ad}(x)\colon L \to L$ *nilpotent.*

Lecture 12.

Theorem 5.4. *(Lie's theorem) Let V be a complex vector space and L a solvable Lie subalgebra of $gl(V)$. Then V has a basis in which every $x \in L$ is represented by an upper-triangular matrix.*

Lemma 5.5. *If $x \in gl(V)$ for V a complex vector space then x has an eigenvector.*

Proposition 5.6. *Let V be a complex vector space and L a solvable Lie subalgebra of $gl(V)$. Then there is a $v \in V$ which is a common eigenvector for all $x \in L$.*

6. Some representation theory

Definition 6.1. A *representation* ψ of a Lie algebra L is a homomorphism $\psi\colon L \to gl(V)$ for some finite-dimensional vector space V.

Definition 6.2. Let L be a Lie algebra. An *L-module* is a vector space V and a map

$$\begin{aligned} L \times V &\to V \\ (x, v) &\mapsto xv \end{aligned}$$

which satisfies

$$\begin{aligned} (\lambda x + \mu y)v &= \lambda(xv) + \mu(yv) \\ x(\lambda v + \mu w) &= \lambda(xv) + \mu(xw) \\ [x, y]v &= x(yv) - y(xv) \end{aligned}$$

for all $x, y \in L$, $v, w \in V$ and $\lambda, \mu \in \mathbb{F}$.

Note 6.1. If $\psi\colon L \to gl(V)$ is a representation then defining $(x, v) \mapsto \psi(x)v$ makes V into an L module. Conversely if V is an L-module defining $\psi(x)\colon V \to V$ by $\psi(x)(v) = xv$ defines a representation.

6.1. Submodules and factor modules.

Definition 6.3. If V is an L-module we say that a subspace W of V is an L-submodule or just a submodule if W is L-invariant. That is if $x(W) \subseteq W$ for all $x \in L$.

Note 6.2. Note that if W is a submodule of V then W is an L-module in its own right.

Example 6.1. Make L into an L module using the adjoint representation. Then a subspace $I \subseteq L$ is a submodule if and only if it is an ideal.

If W is a submodule of an L-module V we can make V/W into an L-module by

$$\begin{aligned} L \times V/W &\to V/W \\ (x, v + W) &\mapsto xv + W \end{aligned}$$

Exercise 6.1. Check this makes V/W into an L-module.

Definition 6.4. The space V/W with the definition as an L-module give above is called a quotient or factor module.

Example 6.2. If I is an ideal then

$$L \times L/I \to L/I$$
$$(x, y + I) \mapsto [x, y] + I$$

makes L/I an L-module.

Lecture 13.

6.2. Irreducible and indecomposable modules.

Definition 6.5. An L-module V is called *irreducible* if the only submodules of V are V and 0.

Example 6.3. If V is 1-dimensional then V is irreducible.

Example 6.4. If L is solvable and V is an irreducible module then V is one-dimensional.

Definition 6.6. If U and W are submodules of an L module V and $V = U \oplus W$ we say that V is the *direct sum* of U and W.

Definition 6.7. An L-module V is called *indecomposable* if we cannot find submodules $U \neq 0 \neq W$ with $V = U \oplus W$.

Note 6.3. Note that if V is irreducible then V is indecomposable but the reverse is not usually the case.

Definition 6.8. An L-module V is called *completely reducible* if it has irreducible submodules $S_1, \dots S_r$ and $V = S_1 \oplus \cdots \oplus S_r$.

Example 6.5. If $L = d(n, \mathbb{F})$ and $S_i = \mathbb{C}e^i$ then $V = \mathbb{C}^n = S_1 \oplus \cdots \oplus S_n$ is completely reducible.

Example 6.6. If $L = b(n, \mathbb{F})$ then each

$$W_i = \{(x_1, x_2, \dots, x_i, 0, \dots, 0) \mid x_1, \dots, x_i \in \mathbb{F}\} \subseteq \mathbb{F}^n$$

is a submodule. In fact these are the only submodules and hence L is indecomposable but not irreducible.

6.3. Homomorphisms.

Definition 6.9. Let V and W be L-modules. A linear map $\theta \colon V \to W$ is an *L-module homomorphism* if $\theta(xv) = x\theta(v)$ for all $x \in L$ and $v \in V$. If θ is bijective it is called an *L-module isomorphism* and we write $V \simeq W$.

Note 6.4. If $\psi_V \colon L \to gl(V)$ and $\psi_W \colon L \to gl(W)$ are two representations then $\theta \colon V \to W$ is an L-module homomorphism if and only if for all $x \in L$ we have $\theta\psi_V(x) = \psi_W(x)\theta$.

Note 6.5. Because $\theta \colon V \to W$ is linear we know that $\ker(\theta)$ and $\operatorname{im}(\theta)$ are subspaces of V and W respectively.

Exercise 6.2. If $\theta \colon V \to W$ is a homomorphism of L-modules show that $\ker(\theta)$ and $\operatorname{im}(\theta)$ are submodules of V and W respectively.

Theorem 6.10. *(Isomorphism theorem) Let L be a Lie algebra.*

a) If $\theta \colon V \to W$ is an L-module homomorphism then $L/\ker(\theta)$ is isomorphic to $\operatorname{im}(\theta)$.
b) If U and W are submodules of and L-module V then $U+W$ and $U \cap W$ are also submodules and $(U+W)/W \simeq U/U \cap W$
c) If U and W are submodules of and L-module V with $V \subseteq U$ then W/U is a submodule of V/U and $(V/U)/(W/U) \simeq V/W$.

Proposition 6.11. *Let W be a submodule of V then the following map is a bijection:*

$$\{U \mid U \text{ is a submodule of } V \text{ and } W \subseteq U\} \to \{S \mid S \text{ is a submodule of } V/W\}$$
$$U \mapsto U/W$$

6.4. **Schur's Lemma.**

Lemma 6.12. *Let L be a complex Lie algebra and S a finite-dimensional, irreducible L-module. A map $\theta\colon S \to S$ is an L-module homomorphism of and only if $\theta = \lambda 1_S$ for some $\lambda \in \mathbb{C}$ where 1_S is the identity map $S \to S$.*

Lemma 6.13. *Let L be a complex Lie algebra and V and irreducible L-module. Let $z \in Z(L)$ then there is a $\lambda \in \mathbb{C}$ such that for all $v \in V$ we have $zv = \lambda v$.*

Lecture 14.

7. Representations of $sl(2, \mathbb{C})$

Definition 7.1. For $d \geq 0$ define V_d to be the complex vector space of all complex, homogeneous polynomials of degree d.

Note 7.1. V_d has basis the monomials $X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d$ and hence has dimension $d+1$.

We define a representation $\psi\colon sl(2,\mathbb{C}) \to gl(V_d)$ by giving the action on the standard basis of $sl(2,\mathbb{C})$ as follows.

$$\psi(e) = X\frac{\partial}{\partial Y}, \quad \psi(f) = Y\frac{\partial}{\partial X} \quad \text{and} \quad \psi(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Note 7.2. The action on a monomial is given by

$$\psi(e)(X^aY^b) = bX^{a+1}Y^{b-1},\ \psi(f)(X^aY^b) = aX^{a-1}Y^{b+1} \quad \text{and} \quad \psi(h)(X^aY^b) = (a-b)X^aY^b.$$

Theorem 7.2. *With the above definition $\psi\colon sl(2,\mathbb{C}) \to gl(V_d)$ is a representation.*

Note 7.3. With respect to the basis $X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d$ the matrices of $\psi(e)$, $\psi(f)$ and $\psi(h)$ are given by

$$\psi(e) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \psi(f) = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d & 0 & \cdots & 0 & 0 \\ 0 & d-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

and

$$\psi(h) = \begin{bmatrix} d & 0 & \cdots & 0 & 0 \\ 0 & d-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -d+2 & 0 \\ 0 & 0 & \cdots & 0 & -d \end{bmatrix}$$

Example 7.1. See handout where $d = 0$, $d = 1$ and $d = 2$ are discussed.

Lecture 15.

Handout: Examples of $sl(2,\mathbb{C})$ representations
Handout: Conventions for matrices

Theorem 7.3. *The $sl(2,\mathbb{C})$ module V_d is irreducible.*

Lemma 7.4. *Suppose V is an $sl(2,\mathbb{C})$ module and $v \in V$ is an eigenvector of h of eigenvalue λ. Then*

(a) $ev = 0$ or ev is an eigenvector of h of eigenvalue $\lambda + 2$
(b) $fv = 0$ or fv is an eivenvector of h of eigenvalue $\lambda - 2$

Lemma 7.5. *Let V be a finite dimensional $sl(2,\mathbb{C})$ module. Then V contains an eigenvector w of h such that $ew = 0$.*

Theorem 7.6. *If V is a finite dimensional irreducible $sl(2,\mathbb{C})$ module then V is isomorphic to one of the V_d.*

Corollary 7.7. *If V is a finite dimensional $sl(2,\mathbb{C})$ module and $w \in V$ is an h eigenvector such that $ew = 0$ then $hw = dw$ for some integer d and the subspace generated by w is isomorphic to V_d.*

Note 7.4. A vector w of the type in the corollary is called a highest weight vector.

Theorem 7.8 (Weyl's Theorem)**.** *Let L be a complex, semisimple Lie algebra. Then every finite-dimensional representation of L is completely reducible.*

Lecture 16.

Handout: Non-degenerate bilinear forms
Handout: Killing form examples

8. Cartan's criterion for semisimplicity

Theorem 8.1 (Jordan decomposition)**.** *If $x \in gl(V)$ for a complex vector space V then there exist unique $d, n \in gl(V)$ such that*

(i) $x = d + n$
(ii) *d is diagonalizable*
(iii) *n is nilpotent*
(iv) $[d, n] = 0$.

Definition 8.2. We call the decomposition $x = d + n$ the Jordan canonical form for x or the Jordan decomposition of x.

Example 8.1.

$$x = \begin{bmatrix} \mu & 1 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad d = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix} \quad n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Lemma 8.3. *Let $x \in gl(V)$ have Jordan decomposition $x = d + n$ then*

(a) *There exists a polynomial $p(t)$ such that $p(x) = d$.*
(b) *Choose a basis for V so that d is diagonal. Define $\bar{d} \in gl(V)$ to be the linear map whose matrix is the complex conjugate to the matrix of d. Then there is a polynomial $q(t)$ such that $q(x) = \bar{d}$.*

Lemma 8.4. *Let $x \in gl(V)$ have Jordan decomposition $x = d + n$ then $\mathrm{ad}(x)$ has Jordan decomposition $\mathrm{ad}(x) = \mathrm{ad}(d) + \mathrm{ad}(n)$.*

Lemma 8.5. *Let V be a complex vector space and $L \subseteq gl(V)$ a Lie subalgebra. Then for all $x \in L'$ and for all $y \in L$ we have $\mathrm{tr}(xy) = 0$.*

Proposition 8.6. *Let V be a complex vector space and $L \subseteq gl(V)$ a Lie subalgebra. If $\mathrm{tr}(xy) = 0$ for all $x, y \in L$ then L is solvable.*

Exercise 8.1. If $x, y, z \in gl(V)$ then $\mathrm{tr}([x, y]z) = \mathrm{tr}(x[y, z])$.

Theorem 8.7. *Let L be a complex Lie algebra. Then L is solvable if and only if $\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ for all $x \in L$ and $y \in L'$.*

Definition 8.8. A *bilinear form* on L is map

$$g \colon L \times L \to \mathbb{F}$$

such that

$$g(\alpha x + \beta y, z) = \alpha g(x, y) + \beta g(y, z)$$
$$g(x, \alpha y + \beta z) = \alpha g(x, y) + \beta g(x, z)$$

for all $x, y, z \in L$ and $\alpha, \beta \in \mathbb{F}$.

Definition 8.9. A bilinear form $g \colon L \times L \to \mathbb{F}$ is *symmetric* if $g(x, y) = g(y, x)$ for all $x, y \in L$.

Definition 8.10. A symmetric, bilinear form $g\colon L \times L \to \mathbb{F}$ is *invariant* if $g(x, [y, z]) = g([x, y], z)$ for all $x, y, z \in L$.

Definition 8.11. Let L be a complex Lie algebra. The Killing form is the map $\kappa\colon L \times L \to \mathbb{F}$ defined by $\kappa(x, y) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(y))$ for all $x, y \in L$.

Lemma 8.12. *The Killing form is bilinear, symmetric and invariant.*

Theorem 8.13 (Cartan's First Criterion)**.** *A complex Lie algebra L is solvable if and only if the Killing form satisfies $\kappa(x, y) = 0$ for all $x \in L$ and $y \in L'$.*

Lecture 17.

Lemma 8.14. *Let I be an idea in a Lie algebra L. Let κ be the Killing form for L and κ_I the Killing form for I. Then if $x, y \in I$ we have $\kappa(x, y) = \kappa_I(x, y)>$*

Definition 8.15. If κ is a symmetric bilinear form on a vector space V and W is a subspace of V we define

$$W^\perp = \{v \in V \mid \kappa(v, w) = 0 \forall w \in W\}.$$

Definition 8.16. A symmetric bilinear form on a vector space V is called *non-degenerate* if $V^\perp = 0$.

Lemma 8.17. *If κ is a non-degenerate, symemtric bilinear form on V and $W \subseteq V$ then* $\dim(W) + \dim(W^\perp) = \dim(V)$*. (Proved in handout.)*

Example 8.2. Consider $\mathbb{C}^2$ with the symmetric bilinear form $(z, w) = z^0 w^0 + z^1 w^1$ and let W be the span of $w = (1, i)$. Then as $(w, w) = 0$ we have $W^\perp = W$ and so it is not true that $W \cap W^\perp = 0$.

Lemma 8.18. *If I is an ideal in a Lie algebra L then $I^\perp$ is an ideal.*

Theorem 8.19 (Cartan's second criterion)**.** *A complex semisimple Lie algebra is semisimply if and only if its Killing form is non-degenerate.*

Lemma 8.20. *If I is an ideal in a complex, semisimple Lie algebra L with $0 \neq I \neq L$ then $L = I \oplus I^\perp$ and I is also semisimple.*

Theorem 8.21. *Let L be a complex Lie algebra. Then L is semisimple if and only if there exist simple ideals $L_1, \ldots, L_r \subset L$ such that $L = L_1 \oplus \cdots \oplus L_r$.*

Lemma 8.22. *If L is semisimple and I is an ideal of L then L/I is semisimple.*

Lecture 18.

8.1. Derivations of semisimple Lie algebras.

Proposition 8.23. *If L is a finite-dimensional complex, semisimple Lie algebras then* $\operatorname{ad}(L) = \operatorname{Der}(L)$.

8.1.1. *Abstract Jordan decomposition.*

Proposition 8.24. *Let L be complex Lie algebra. Suppose δ is a derivation of L and $\delta = \sigma + \nu$ is its Jordan decomposition then σ and ν are derivations of L.*

Theorem 8.25. *Let L be a complex, semisimple Lie algebra. Then each $x \in L$ can be written uniquely as $x = d + n$ where $\operatorname{ad}(x)$ is diagonalizable, $\operatorname{ad}(n)$ is nilpotent and $[d, n] = 0$. Moreover if $[y, x] = 0$ then $[y, d] = [y, n] = 0$.*

Definition 8.26. Let $x \in L$ a complex, semisimple Lie algebra. The decomposition $x = d + n$ in the preceding theorem is called the *abstract Jordan decomposition* of x.

Theorem 8.27. *Let L be a complex semisimple Lie algebra and $\theta\colon L \to gl(V)$ be a representation. Suppose x has abstract Jordan decomposition $x = d + n$ then the Jordan decomposition of $\theta(x)$ is $\theta(x) = \theta(d) + \theta(n)$. (Proof omitted.)*

9. The root space decomposition

Example 9.1. Consider $sl(n,\mathbb{C})$ and let $H \subset sl(n,\mathbb{C})$ be the subalgebra of diagonal matrices. If h is a diagonal matrix with ith diagonal element h_i let $\epsilon\colon H \to \mathbb{C}$ be the map $\epsilon_i(h) = h_i$. We have $[h, E_{ij}] = (h_i - h_j)E_{ij} = (\epsilon_i - \epsilon_j)(h)E_{ij}$. Thus $\epsilon_i - \epsilon_j$ is a weight of H with weight space

$$L_{ij} = \operatorname{span}(E_{ij})$$

and

$$sl(n,\mathbb{C}) = H \oplus \bigoplus_{i \neq j} L_{ij}.$$

Definition 9.1. Let $x \in L$ a complex, semisimple Lie algebra. Then x is called *semisimple* if the abstract Jordan decomposition of x is $x = d + n$ with $n = 0$.

Definition 9.2. Let L be a complex, semisimple Lie algebra and $H \subset L$ an abelian Lie algebra all of whose elements are semisimple. Let Φ be the set of all non-zero weights $\alpha \in H^*$ of H. That is $\alpha \in \Phi$ if and only if

$$L_\alpha = \{x \in L \mid [h,x] = \alpha(h)x\} \neq 0.$$

As we can simultaneously diagonalise all the elements of H (see appendix of textbook), we have

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

where

$$L_0 = C_L(H) = \{x \in L \mid [h,x] = 0 \forall h \in H\}.$$

Lemma 9.3. *In the situation above if $\alpha, \beta \in H^*$ then*

(a) $[L_\alpha, L_\beta] = L_{\alpha+\beta}$.
(b) *If $\alpha + \beta \neq 0$ then $\kappa(L_\alpha, L_\beta) = 0$, that is if $x \in L_\alpha$ and $y \in L_\beta$ then $\kappa(x,y) = 0$.*
(c) *The restriction of the Killing form to L_0 is non-zero.*

Lecture 19.

9.1. Cartan subalgebras.

Definition 9.4. Let L be a complex, semisimple Lie algebra. A Lie subalgebra $H \subseteq L$ is called a *Cartan subalgebra* of L (CSA) if it is abelian, all its elements are semisimple and it is maximal with respect to these two properties.

Note 9.1. Being maximal means that if H' is abelian and has all its elements semisimple and $H \subset H'$ then $H = H'$.

Note 9.2. While Cartan subalgebras are not unique it turns out that their dimension is always the same. This is called the *rank* of L

Example 9.2. The diagonal matrices in $sl(n,\mathbb{C})$ form a Cartan subalgebra.

Example 9.3. If g is an n by n complex matrix of determinant one then the subalgebra of $sl(n,\mathbb{C})$ consisting of all matrices X for which gXg^{-1} is diagonal is a Cartan subalgebra.

Proposition 9.5. *If H is a Cartan subalgebra in a complex, semisimple Lie algebra L then $C_L(H) = H$.*

9.2. Root space decomposition.

Definition 9.6. If H is a CSA the weight space decomposition becomes

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

We call elements of Φ *roots* and the L_α root spaces.

9.3. Subalgebras isomorphic to $sl(2,\mathbb{C})$.

Lemma 9.7. *Suppose $\alpha \in \Phi$ and $x \in L_\alpha$ with $x \neq 0$. Then $-\alpha \in \Phi$ and there is a $y \in L_{-\alpha}$ such that span$\{x, y, [x, y]\}$ is a Lie subalgebra of L isomorphic to $sl(2,\mathbb{C})$.*

Proposition 9.8. *Let V be a complex vector space and $x, y\colon VtoV$ linear maps such that $[x,[x,y]] = 0 = [y,[x,y]]$. Then $[x, y]$ is nilpotent.*

Note 9.3. Given $\alpha \in \Phi$ and x and y as in the Lemma we let $e_\alpha = x$ and rescale y to get f_α such that $h_\alpha = [e_\alpha, f_\alpha]$ satisfies $\alpha(h_\alpha) = 2$. Then $h \mapsto h_\alpha$, $e \mapsto e_\alpha$ and $f \mapsto f_\alpha$ defines an isomorphism from $sl(2,\mathbb{C})$ to span$e_\alpha, h_\alpha, f_\alpha$. We denote span$e_\alpha, h_\alpha, f_\alpha$ to $sl(\alpha)$.

9.4. Root strings and eigenvalues.

Note 9.4. Define $\chi\colon H \to H^*$ by $\chi(h)(k) = \kappa(h,k)$ for all $h, k \in H$. Then χ is an isomorphism. Define $t_\alpha \in H$ by $\chi(t_a) = \alpha$ or $k(t_\alpha, k) = \alpha(k)$ for all $k \in K$.

Lemma 9.9. *Let $\alpha \in \Phi$. If $x \in L_\alpha$ and $y \in L_\alpha$ then $[x, y] = \kappa(x, y)t_\alpha$. In particular h_α is a multiple of t_α>*

Lemma 9.10. *If $M \subseteq L$ is an $sl(\alpha)$ submodule then the eigenvalues of h_α acting on M are integers.*

Lecture 20.

Proposition 9.11. *Let $\alpha \in \Phi$. Then $\dim(L_{\pm\alpha}) = 1$ and $n\alpha \in \Phi$ if and only if $n = \pm 1$.*

Proposition 9.12. *Suppose $\alpha, \beta \in \Phi$, $\beta = \pm\alpha$.*

(a) $\beta(h_\alpha) \in \mathbb{Z}$
(b) $\exists r, q \in \mathbb{Z}$ *such that if $k \in \mathbb{Z}$ then $\beta + k\alpha \in \Phi$ if and only if $-r \le k \le q$ and $r - q \in \beta(h_\alpha)$*>
(c) $\beta - \beta(h_\alpha)\alpha \in \Phi$.

9.5. Cartan subalgebra as an inner product space.

Lemma 9.13.

(i) If $h \in H$ and $h \neq 0$ then $\exists \alpha \in H^*$ such that $\alpha(h) \neq 0$.
(ii) span$(\Phi) = H^*$

Lemma 9.14. *For each $\alpha \in \Phi$.*

(1)
$$t_\alpha = \frac{h_\alpha}{\kappa(e_\alpha, f_\alpha)} \quad \text{and} \quad h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}.$$
(2)
$$\kappa(t_\alpha, t_\alpha)\kappa(h_\alpha, h_\alpha) = 4.$$

Lecture 21.

Corollary 9.15. *If α and β are roots then $\kappa(h_\alpha, h_\beta) \in \mathbb{Z}$ and $\kappa(t_\alpha, t_\beta) \in \mathbb{Q}$.*

Lemma 9.16. *If $\alpha_1, \dots, \alpha_r$ is a basis of H^* made up of roots and β is a root then $\beta = \sum_{i=1}^r q_i\alpha_i$ with $q_i \in \mathbb{Q}$.*

Proposition 9.17. *If $\alpha_1, \dots, \alpha_r$ is a basis of H^* made up of roots then*
$$span_{\mathbb{R}}\{\alpha \mid \alpha \in \Phi\} = span_{\mathbb{R}}\{\alpha_1, \dots, \alpha_r\}$$
where span$_{\mathbb{R}}$ means the real span.

Definition 9.18. Define a real vector space E by
$$E = \mathrm{span}_{\mathbb{R}}\{\alpha \mid \alpha \in \Phi\}.$$

Definition 9.19. We define a bilinear symmetric form $(\ ,\ )$ on H^* by making it equal to the Killing form under the isomorphism χ. In other words $(\alpha, \beta) = \kappa(t_\alpha, t_\beta)$.

Proposition 9.20. *The bilinear symmetric form $(\ ,\ )$ on E is an inner product.*

10. Root systems

Note 10.1. Let E be a finite, dimensional real vector space with inner product $(\ ,\ )$. If $0 \neq v \in E$ recall that s_v, the reflection in the hyperspace orthogonal to v satisfies

$$s_v(x) = x - \frac{2(x,v)}{(v,v)}x$$

for all $x \in X$. We define

$$\langle x, v\rangle = \frac{2(x,v)}{(v,v)}$$

for all $x, v \in V$.

Definition 10.1. A subset R of a real inner product space E is called a *root system* if

(R1) R is finite, spans E and $0 \notin R$.
(R2) If $\alpha \in R$ then $\lambda\alpha \in R$ if and only if $\alpha = \pm 1$.
(R3) If $\alpha \in R$ then $s_\alpha(R) = R$.
(R4) If $\alpha, \beta \in R$ then $\langle \beta, \alpha\rangle \in \mathbb{Z}$.

Note 10.2. The elements of R are called *roots*.

Proposition 10.2. *If L is a complex semisimple Lie algebra and H is a Cartan subalgebra with roots $\Phi \subset H^*$ then $\Phi \subset E$ is a root system.*

Note 10.3. It turns out that every root system arises in this way.

Example 10.1. See the handout on $sl(n, \mathbb{C})$.

Lemma 10.3 (Finiteness Lemma). *Suppose R is a root system. Then if $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$ then*

$$\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle \in \{0, 1, 2, 3\}.$$

Note 10.4. Assume $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$ and $(\beta, \beta) \geq (\alpha, \alpha)$. Then α, β must satisfy one of the following:

$\langle\alpha,\beta\rangle$	$\langle\beta,\alpha\rangle$	$\cos(\theta)$	θ	$\frac{\langle\beta,\beta\rangle}{\langle\alpha,\alpha\rangle}$
0	0	0	$\pi/2$	indeterminate
1	1	1/2	$\pi/3$	1
-1	-1	-1/2	$2\pi/3$	1
1	2	$1/\sqrt{2}$	$\pi/4$	2
1	-2	$-1/\sqrt{2}$	$3\pi/4$	2
1	3	$\sqrt{3}/2$	$\pi/6$	3
1	-3	$-\sqrt{3}/2$	$5\pi/6$	3

Proposition 10.4. *Let $\alpha, \beta \in R$ then*

(a) If the angle between α and β is strictly obtuse then $\alpha + \beta \in R$.
(b) If the angle between α and β is strictly acute then $\alpha - \beta \in R$.

Lecture 22.

Handout: Root system of $sl(n, \mathbb{C})$
Handout: Two-dimensional root systems
Handout: Dynkin diagrams

Definition 10.5. An *isomorphism* between two root systems $R \subset E$ and $R' \subset E'$ is a linear isomorphism $\varphi\colon E \to E'$ such that

(a) $\varphi(R) = R'$
(b) $\forall\, \alpha, \beta \in R$ we have $\langle \alpha, \beta\rangle = \langle \varphi(\alpha), \varphi(\beta)\rangle$

Note 10.5. This shows that if $R \subset E$ and we change the inner product on E by multiplying it by a positive constant then R is still a root system isomorphic to the original R.

Note 10.6. We will not prove it but if we vary the Cartan subalgebra of a complex, semisimple Lie algebra then the root systems that arise are all isomorphic.

Example 10.2. Handout about two-dimensional root systems.

Definition 10.6. A root system R is called *irreducible* if it cannot be written as a disjoint union $R = R_1 \cup R_2$ where neither R_1 or R_2 is empty and $(\alpha_1, \alpha_2) = 0$ for all $\alpha_1 \in R_1$ and $\alpha_2 \in R_2$.

Lemma 10.7. *Let R be a root system then we can write it as a disjoint union $R = R_1 \cup \cdots \cup R_k$ where each R_i is an irreducible root system in $E_i = span(R_i)$ and E is the orthogonal direct sum of the E_i.*

Note 10.7. A Lie algebra is simple if and only if its root system is irreducible. Moreover if $L = L_1 \oplus \cdots \oplus L_k$ where each L_i is simple then the root system R of L can be written as a disjoint union $R = R_1 \cup \cdots \cup R_k$, as in the Lemma above, and R_i is the root system of L_i.

10.1. **Bases for root systems.**

Definition 10.8. If R is a root system a subset $B \subset R$ is called a *base* for R if

(B1) B is a basis for E.
(B2) For every $\beta \in R$ we have $b = \sum_{\alpha \in B} k_\alpha \alpha$ where $k_\alpha \in \mathbb{Z}$ and every non-zero coefficient k_α has the same sign.

Proposition 10.9. *Every root system has a base.*

Note 10.8. Although we won't prove it every base arises by the construction in the proof of the proposition.

Note 10.9. Bases are not unique but as they form a basis for E they must all have the same number of elements.

Definition 10.10. We call the elements of a base for a root system *simple roots.*

Lecture 23.

Definition 10.11. If we have chosen a base B for a root system then the non-zero roots of the form $b = \sum_{\alpha \in B} k_\alpha \alpha$ with every k_α non-negative are called *positive* and denoted R^+.

Definition 10.12. The Weyl group of a root system is the group generated by the root reflections.

Lemma 10.13. *The Weyl group is finite.*

Theorem 10.14. *If B and B' are two bases for a root system R then there is a unique element of the Weyl group w such that $w(B) = B'$. Moreover if B is a base then $w(B)$ is also a base for any w in the Weyl group. (Proof omitted.)*

10.2. **Cartan matrix and Dynkin diagram of a root system.**

Definition 10.15. Let R be a root system with base $B = \{\alpha_1, \dots, \alpha_\ell\}$. Then the Cartan matrix $C = [C_{ij}]$ of R with respect to B is defined by $C_{ij} = \langle \alpha_i, \alpha_j \rangle$.

Note 10.10. Notice that the Cartan matrix is unique up to conjugation by any permutation of the labels of the elements of the base. It also follows from the fact that all bases are related by a Weyl group transformation that when we change the base the Cartan is conjugated by the corresponding permutation.

Definition 10.16. Let R be a root system with base $B = \{\alpha_1, \dots, \alpha_\ell\}$. Then the Dynkin diagram of R with respect to the base B has a node for every simple root and the nodes are joined by

$$d_{ij} = C_{ij} C_{ji} = \langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$$

edges. If $d_{ij} > 1$ we put an arrow on the edges between nodes α_i and α_j pointing in the direction of the smaller simple root.

Note 10.11. Clearly we have $C_{ii} = 2$. If B is a base and $\alpha, \beta \in B$ then $(\alpha, \beta) \leq 0$ as otherwise $\alpha - \beta$ is a root which is not possible. So $C_{ij} \leq 0$ for all i, j.

Theorem 10.17. *A complex semisimple Lie algebra is determined by its Dynkin diagram.*

Theorem 10.18 (Serre's Theorem)**.** *Let C be the Cartan matrix of a root system of rank. Define a Lie algebra with generators e_i, h_i and f_i for $i = 1, \dots, r$ where r is the size of the Cartan matrix, subject to the relations*

(S1) $[h_i, h_j] = 0 \forall i, j$
(S2) $[h_i, e_j] = c_{ji} e_j$ *and* $[h_i, e_j] = -c_{ji} e_j$ *for all* i, j
(S3) $[e_i, f_i] = h_i \forall i$ *and* $[e_i, f_j] = 0 \forall i \neq j$
(S4) $(\mathrm{ad}(e_i))^{1-C_{ji}}(e_j) = 0$ *and* $(\mathrm{ad}(e_f))^{1-C_{ji}}(e_f) = 0$ *if* $i \neq j$.

Then L is a finite dimensional, complex semisimple Lie algebra with Cartan subalgebra spanned by the $h_1, \dots, h_r$ and Cartan matrix C.