Lie Algebras IV 2008

Comments on the mock exam

I won't give complete solutions because much of this was covered in lectures.

SO THESE ARE NOT MODEL SOLUTIONS.

1. I don't expect lengthy answers. Examples are below.

(i) True. Constructed in lectures.
(ii) False. It is the other way around. For example the upper triangular matrices are solvable but not nilpotent.
(iii) True. Cartan's criterion.
(iv) False. The image of the adjoint map is an ideal.
(v) True. There exists an irreducible representation of $sl(2,\mathbb{C})$ of any dimension.
(vi) True. Proved in lectures or fundamental theorem of algebra shows there is a root λ of $\det(X - xI)$ and thus $X - \lambda I$ has non-zero kernel which gives the required vector v.
(vii) True. Follows from definition of ideal that $[I_1, I_2] \subset I_1$ and also $[I_1, I_2] \subset I_2$.
(viii) False. There are lots of irreducibles of different dimensions which cannot be isomorphic.
(ix) True. Proved in lectures.
(x) False. The inner product of any two simple roots is negative.

2. (a) Lectures.

(b) This was constructed in class. To show that it was an actual Lie algebra it is simplest to represent it as the matrix algebra in the next part of the question.

(c)

$$L' = [L, L] = \left\{ \begin{bmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid y \in \mathbb{C} \right\}$$

and $Z(L) = L'$.

(i) L is soluble as $[L', L'] = [Z(L), Z(L)] = 0$.

(ii) L is nilpotent as $[L, L'] = [L, Z(L)] = 0$.

3. (a) (i) Lectures.

(ii) Assume $L^k = 0$ and $L^{k-1} \neq 0$. Then $[L, L^{k-1}] = L^k = 0$. So $0 \neq L^{k-1} \subset Z(L)$.

(iii) We have $[L + Z(L), L + Z(L)] = [L, L] + Z(L)$ and by induction $[L + Z(L), L^k + Z(L)] = L^k + Z(L)$. So if $L/Z(L)$ is nilpotent there must be an m such that $L^m \subset Z(L)$. But then $L^{m+1} = [L, L^m] \subset [L, Z(L)] = 0$ so L is nilpotent.

(iv) No. For example let L be the non-abelian two-dimensional Lie algebra with basis $\{x, y\}$ and satisfying $[x, y] = x$. Let I be the span of x. Then $[L, L] = I$ and $[L, I] = I$ so $L^k = I$ for all k and L is not nilpotent. But L/I is one-dimensional so abelian and thus nilpotent. Likewise I is abelian and nilpotent.

(b) (i) Note first that if D is a derivation then $D \circ \mathrm{ad}(x)(y) = D([x, y]) = [D(x), y] + [x, D(y)] = \mathrm{ad}(D(x))(y) + \mathrm{ad}(x) \circ D(y)$ so that $[D, \mathrm{ad}(x)] = \mathrm{ad}(D(x))$. So if $D \in \mathrm{Der}(L)$ and $\mathrm{ad}(x) \in \mathrm{IDer}(L)$ then $[D, \mathrm{ad}(x)] = \mathrm{ad}(D(x)) \in \mathrm{IDer}(L)$ and thus $\mathrm{IDer}(L)$ is an ideal in $\mathrm{Der}(L)$.

(ii) Consider $\mathrm{ad}\colon L \to \mathrm{Der}(L)$. The image is $\mathrm{IDer}(L)$ so we have that $\mathrm{IDer}(L) = L/Z(L)$ from the the isomorphism theorem. We have already proved that if $L/Z(L)$ is nilpotent so is L. On the other hand from the same answer to that question $(L + Z(L))^k = L^k + Z(L)$ so that if $L^k = 0$ for some k then $(L + Z(L))^k = 0$ and so $\mathrm{IDer}(L)$ is nilpotent.

(iii) From the identity above $[D, x] = \mathrm{ad}(D(x))$ so we have $\mathrm{ad}(D(x) = 0$ and hence $D(x) \in \ker(\mathrm{ad}) = Z(L)$ so that $D(L) \subset Z(L)$.

4. (a) (i) Lectures.

(ii) $\mathrm{ad}\colon L \to gl(L)$.

(iii) $(x, [y, z]) = ([x, y], z)$ for all $x, y, z \in L$.

(iv) $(x, [y, z])_f = \mathrm{tr}(f(x)f([y, z])) = \mathrm{tr}(f(x)[f(y), f(z)]) = \mathrm{tr}(f(x)f(y)f(z)) - \mathrm{tr}(f(x)f(z)f(y)) = \mathrm{tr}(f(x)f(y)f(z)) - \mathrm{tr}(f(y)f(x)f(z)) = \mathrm{tr}([f(x), f(y)]f(z)) = \mathrm{tr}(f([x, y])f(z)) = ([x, y], z)_f$.

(b) This is a calculation we have done in a handout. $\kappa(h, e) = 0 = \kappa(h, f)$ and $\kappa(e, f) = 4$.

(c) Take H as the span of h. The span of e is a root space for the root α defined by $\alpha(h) = 2$ and the span of f is the root space $-\alpha$.

5. (a) (i) Lectures

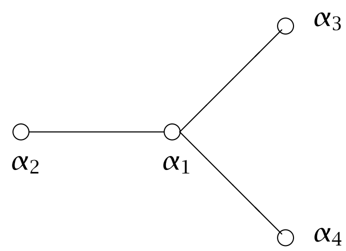
(ii) Lectures

(b) (i) If there are p positive roots there are p negative roots and therefore $2p$ roots. From the root space decomposition and the fact that the dimension of the root spaces are one and the dimension of the Cartan subalgebra is r we see that the dimension of L is $2p + r$.

(ii) Lectures

(iii) Lectures

(c) Notice that everything is joined to α_1 and none of α_2, α_3 and α_4 are joined to each other. All bonds are single so the Dynkin diagram must be



(d) The diagonal entries of the Cartan matrix are always 2. The only bonds are joining α_2 to each of α_1, α_3 and α_4 so that $C_{13} = C_{31} = 0$, $C_{14} = C_{41} = 0$, $C_{34} = C_{43} = 0$ so the basic shape of the Cartan matrix is

$$\begin{bmatrix} 2 & * & 0 & 0 \\ * & 2 & * & * \\ 0 & * & 2 & 0 \\ 0 & * & 0 & 2 \end{bmatrix}$$

All the bonds are single so we must have

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$