Lie Algebras IV 2009

Killing form examples.

1. Killing form of $sl(2,\mathbb{C})$

Recall from the handout about $sl(2,\mathbb{C})$ representations that the matrices for the adjoint representation are:

$$\mathrm{ad}(e)=\begin{bmatrix}0&-2&0\\0&0&1\\0&0&0\end{bmatrix},\quad \mathrm{ad}(f)=\begin{bmatrix}0&0&0\\-1&0&0\\0&2&0\end{bmatrix}\quad\text{and}\quad \mathrm{ad}(h)=\begin{bmatrix}2&0&0\\0&0&0\\0&0&-2\end{bmatrix}$$

From this we can calculate that

$$\kappa(h,h)=8,\kappa(e,e)=\mathrm{tr}\begin{bmatrix}0&0&-2\\0&0&0\\0&0&0\end{bmatrix}=0\quad\text{and}\quad\kappa(f,f)=\mathrm{tr}\begin{bmatrix}0&0&0\\0&0&0\\-2&0&0\end{bmatrix}=0$$

and

$$\kappa(e,f)=\mathrm{tr}\begin{bmatrix}2&0&0\\0&2&0\\0&0&0\end{bmatrix}=4,\kappa(e,h)=\mathrm{tr}\begin{bmatrix}0&0&0\\0&0&-2\\0&0&0\end{bmatrix}=0,\quad\text{and}\quad\kappa(f,h)=\mathrm{tr}\begin{bmatrix}0&0&0\\-2&0&0\\0&0&0\end{bmatrix}=0.$$

We check non-degeneracy. Assume that $x=\alpha e+\beta h+\gamma f$ is in L^o that is $\kappa(x,y)=0$ for all $y\in L$. Then $0=\kappa(x,e)=4\gamma$, $0=\kappa(x,h)=8\alpha$ and $0=\kappa(x,f)=4\alpha$ so that $x=0$.

2. Killing form of Heisenberg Lie algebra

Recall that we have

$$f=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\ g=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},\quad\text{and}\quad z=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

with $[f,g]=z\in Z(L)$ and all other commutators being zero. Taking $\{f,z,g\}$ as a basis of L we obtain

$$\mathrm{ad}(f)=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix},\quad\mathrm{ad}(z)=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\quad\text{and}\quad\mathrm{ad}(g)=\begin{bmatrix}0&0&0\\-1&0&0\\0&0&0\end{bmatrix}$$

Hence $\kappa(z,x)=0$ for any $x\in L$ and, moreover, $\kappa(x,y)=0$ for any $x,y\in L$. Hence $L^0=L$ and we deduce the fact that L is solvable. We know this because $[L,L]=z\mathbb{C}$ and hence $L^{(2)}=0$.