Lie Algebras IV 2009

Comments on 2008 exam

I won't give complete solutions because much of this was covered in lectures.

SO THESE ARE NOT MODEL SOLUTIONS.

1. I don't expect lengthy answers. Examples are below.

(i) False. If $L = \mathbb{C}x$ then $[\lambda x, \mu x] = \lambda\mu[x, x] = 0$ for all $\lambda, \mu \in \mathbb{C}$ so that L is abelian.
(ii) True. Lectures.
(iii) True. Cartan's criterion says it is non-degenerate.
(iv) False. Proved in lectures that the kernel of a homomorphism is an ideal.
(v) False. There exists an irreducible representation of $sl(2, \mathbb{C})$ of any dimension.
(vi) True. Proved in lectures or fundamental theorem of algebra shows there is a root λ of $\det(X - xI)$ and thus $X - \lambda I$ has non-zero kernel which gives the required vector v.
(vii) True. Follows from definition of ideal that $[I_1, I_2] \subset I_1$ and also $[I_1, I_2] \subset I_2$.
(viii) False. They would need to have the same dimension to be isomomorphic.
(ix) True. Proved in lectures.
(x) False. The inner product of any two simple roots is negative.

2. (a) (i) Lectures.

(ii) Lectures.

(iii) Lectures. Just have to check that if D and E are derivations then so also is $[D, E] = DE - ED$.

(iv) This follows from the fact proved in class that if D is a derivation then $[D, \mathrm{ad}(x)] = \mathrm{ad}(D(x))$. A condition is that L is semisimple.

(b) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

3. (a) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

(iv) Lectures.

(v) Consider $\mathrm{ad}(L) \subset \mathrm{gl}(L)$. By (iii) every element of $\mathrm{ad}(L)$ is nilpotent. So by Engel's Theorem there is a basis $\{v_1, \ldots, v_r\}$ of L such that each $\mathrm{ad}(x)$ is strictly upper triangular. In particular $\mathrm{ad}(x)(v_1) = 0$ for all $x \in L$. As v_1 is a basis element it cannot be zero so we have that $\mathbb{C}v_1$ is a one-dimensional ideal.

(b) (i) Lectures.

(ii) Particular case of general result proved in lectures. What I wanted was the proof of the general result adapted to this case.

(iii) The Lie algebra of all upper-triangular matrices is solvable but not nilpotent.

4. (a) (i) Lectures.

(ii) Lectures.

(iii) Assignment.

(b) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

(iv) If $x \in Z(L)$ then $\mathrm{ad}(x) = 0$ so that $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0$ for any $y \in L$. But L is non-degenerate so that $x = 0$.

5. (a) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

(b) (i) Lectures

(ii) F_4 – the Dynkin diagram is in the notes.

(ii) C_3 – the Cartan matrix is in the notes. **6.**