

Lie Algebras IV 2009
Comments on 2008 exam

I won't give complete solutions because much of this was covered in lectures.

SO THESE ARE NOT MODEL SOLUTIONS.

1. I don't expect lengthy answers. Examples are below.

- (i) False. If $L = \mathbb{C}x$ then $[\lambda x, \mu x] = \lambda\mu[x, x] = 0$ for all $\lambda, \mu \in \mathbb{C}$ so that L is abelian.
- (ii) True. Lectures.
- (iii) True. Cartan's criterion says it is non-degenerate.
- (iv) False. Proved in lectures that the kernel of a homomorphism is an ideal.
- (v) False. There exists an irreducible representation of $sl(2, \mathbb{C})$ of any dimension.
- (vi) True. Proved in lectures or fundamental theorem of algebra shows there is a root λ of $\det(X - xI)$ and thus $X - \lambda I$ has non-zero kernel which gives the required vector v .
- (vii) True. Follows from definition of ideal that $[I_1, I_2] \subset I_1$ and also $[I_1, I_2] \subset I_2$.
- (viii) False. They would need to have the same dimension to be isomorphic.
- (ix) True. Proved in lectures.
- (x) False. The inner product of any two simple roots is negative.

2. (a) (i) Lectures.

(ii) Lectures.

(iii) Lectures. Just have to check that if D and E are derivations then so also is $[D, E] = DE - ED$.

(iv) This follows from the fact proved in class that if D is a derivation then $[D, \text{ad}(x)] = \text{ad}(D(x))$. A condition is that L is semisimple.

(b) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

3. (a) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

(iv) Lectures.

(v) Consider $\text{ad}(L) \subset \text{gl}(L)$. By (iii) every element of $\text{ad}(L)$ is nilpotent. So by Engel's Theorem there is a basis $\{v_1, \dots, v_r\}$ of L such that each $\text{ad}(x)$ is strictly upper triangular. In particular $\text{ad}(x)(v_1) = 0$ for all $x \in L$. As v_1 is a basis element it cannot be zero so we have that $\mathbb{C}v_1$ is a one-dimensional ideal.

(b) (i) Lectures.

(ii) Particular case of general result proved in lectures. What I wanted was the proof of the general result adapted to this case.

(iii) The Lie algebra of all upper-triangular matrices is solvable but not nilpotent.

4. (a) (i) Lectures.

(ii) Lectures.

(iii) Assignment.

(b) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

(iv) If $x \in Z(L)$ then $\text{ad}(x) = 0$ so that $\kappa(x, y) = \text{tr}(\text{ad}(x) \text{ad}(y)) = 0$ for any $y \in L$. But L is non-degenerate so that $x = 0$.

5. (a) (i) Lectures.

(ii) Lectures.

(iii) Lectures.

(b) (i) Lectures

(ii) F_4 - the Dynkin diagram is in the notes.

(ii) C_3 - the Cartan matrix is in the notes. **6.**