THE UNIVERSITY OF ADELAIDE
AUSTRALIA

Examination in School of Mathematical Sciences

Semester 1, 2008

016676	**Honours Pure Mathematics — Lie Algebras IV — MOCK EXAM**
	PURE MTH 4005A

Official Reading Time:	10 mins
Writing Time:	<u>180 mins</u>
Total Duration:	190 mins

NUMBER OF QUESTIONS: 5 TOTAL MARKS: 100

Instructions

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

Materials

- 1 Blue book is provided.
- Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

Each question is worth approximately the same number of marks.
In this exam all Lie algebras and vector spaces will be finite dimensional over the complex numbers $\mathbb{C}$.

1. Answer each of the following questions true or false. Give a short justification in each case. This might just be 'result proved in lectures' or the name of a theorem.
   (i) There exists a two-dimensional non-abelian Lie algebra.
   (ii) A solvable Lie algebra has to be nilpotent.
   (iii) The Killing form of a semisimple Lie algebra is non-degenerate.
   (iv) The image of a homomorphism of Lie algebras is never an ideal.
   (v) There exists an irreducible representation of $sl(2, \mathbb{C})$ of dimension 987654321.
   (vi) If V is a complex vector space and $X \colon V \to V$ is a linear map then there is a vector $v \in \mathbb{C}$ with $v \neq 0$ and $Xv = \lambda v$ for some $\lambda \in \mathbb{C}$.
   (vii) If I_1 and I_2 are ideals in a Lie algebra L then $[I_1, I_2] \subset I_1 \cap I_2$.
   (viii) Any two representations of $sl(2, \mathbb{C})$ are isomorphic.
   (ix) Root spaces of a complex, semisimple Lie algebra are always one-dimensional.
   (x) There is a Lie algebra L with Killing form $\kappa(\ ,\ )$ and base of simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $(\alpha_1, \alpha_3) = 0$.

2. (a) (i) Define what it means for a vector space L to be a Lie algebra.
   (ii) Define what it means for a subset I of a Lie algebra L to be an ideal.
   (iii) Define the derived algebra $L' = [L, L]$ of L and prove that it is an ideal.
   (b) Determine the 3-dimensional complex Lie algebras L for which $L' = [L, L] = \operatorname{span}\{x\}$ is one-dimensional and $\operatorname{span}\{x\} \subseteq Z(L)$.
   (c) For the Lie algebra
   $$L = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} | \alpha, \beta, \gamma \in \mathbb{C} \right\}$$
   of 3×3 matrices under commutation $[\quad]$, find $[L, L]$ and $Z(L)$ and hence determine if L is
   (i) soluble
   (ii) nilpotent.

Please turn over for page 3

3. (a) (i) Define what it means for a Lie algebra to be nilpotent.

 (ii) Using the definition show that if L is nilpotent then $Z(L) \neq 0$.

 (iii) Show that if $L/Z(L)$ is nilpotent then L is nilpotent.

 (iv) Does the result in (iii) result hold in general? That is, if I is a nilpotent ideal and L/I is also nilpotent is L necessarily nilpotent? Give a reason for your answer.

 (b) Let L be a Lie algebra, $\mathrm{Der}(L)$ the Lie algebra of derivations of L and $\mathrm{IDer}(L)$ the inner derivations of L.

 Prove that:

 (i) $\mathrm{IDer}(L)$ is an ideal of $\mathrm{Der}(L)$;

 (ii) L is nilpotent if and only if $\mathrm{IDer}(L)$ is nilpotent;

 (iii) if $D \in \mathrm{Der}(L)$ and $[D, x] = 0 \quad \text{for all} \quad x \in L$ then $D(L) \subseteq Z(L)$.

4. (a) Let L be a Lie algebra.

 (i) Define what a representation of L is.

 (ii) Prove that every Lie algebra has a representation.

 (iii) Define what it means for a bilinear form on L to be invariant ?

 (iv) If $f\colon L \to gl(V)$ is a representation show that $(\ ,\ )_f$ defined by $(x, y)_f = \mathrm{tr}(f(x)f(y))$ is invariant. Recall that $gl(V)$ denotes the Lie algebra of all linear transformations from V to V.

 (b) Let $L = s\ell_2(\mathbb{C})$ be the algebra of all 2 by 2 matrices with zero trace. Recall that $sl(2, \mathbb{C})$ has basis

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

 (i) By calculation find $\kappa(h, e), \kappa(h, f)$ and $\kappa(e, f)$ where $\kappa(\ ,\ )$ is the Killing form.

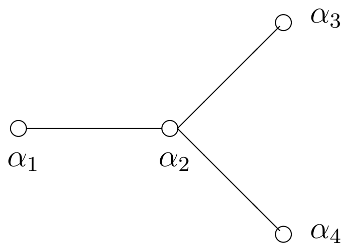
 (ii) Give a Cartan subalgebra of $s\ell_2(\mathbb{C})$ and derive the corresponding root space decomposition.

5. (a) Let L be a semisimple Lie algebra.
   (i) Define what is meant by $H \subset L$ being a Cartan subalgebra and what the corresponding root space decomposition of L is.
   (ii) If roots α and β satisfy $\alpha + \beta \neq 0$ and $x \in L_\alpha$ and $y \in L_\beta$ show that $\kappa(x, y) = 0$ where the L_α and L_β are root spaces and $\kappa(\ ,\ )$ is the Killing form.

   (b) Let H be a Cartan subalgebra for L and $\{\alpha_1, \ldots, \alpha_r\}$ a base of simple roots for L.
   (i) If there are p positive roots what is the dimension of L ?
   (ii) Define the Cartan matrix of L.
   (iii) Define the Dynkin diagram of L.

   (c) If $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a base of simple roots with Cartan matrix $\begin{bmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix}$,
   draw the corresponding Dynkin diagram.

   (d) Suppose the semi-simple Lie algebra L has Dynkin diagram



   where $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a base of simple roots.
   Find the corresponding Cartan matrix.