Examination in School of Mathematical Sciences
Semester 1, 2008

016676 Honours Pure Mathematics — Lie Algebras IV — MOCK EXAM
PURE MTH 4005A

Official Reading Time: 10 mins
Writing Time: 180 mins
Total Duration: 190 mins

NUMBER OF QUESTIONS: 5       TOTAL MARKS: 100

Instructions

• Attempt all questions.
• Begin each answer on a new page.
• Examination materials must not be removed from the examination room.

Materials

• 1 Blue book is provided.
• Calculators are not permitted.

DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.

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Each question is worth approximately the same number of marks.

In this exam all Lie algebras and vector spaces will be finite dimensional over the complex numbers $\mathbb{C}$.

1. Answer each of the following questions true or false. Give a short justification in each case. This might just be ‘result proved in lectures’ or the name of a theorem.
   (i) There exists a two-dimensional non-abelian Lie algebra.
   (ii) A solvable Lie algebra has to be nilpotent.
   (iii) The Killing form of a semisimple Lie algebra is non-degenerate.
   (iv) The image of a homomorphism of Lie algebras is never an ideal.
   (v) There exists an irreducible representation of $sl(2,\mathbb{C})$ of dimension 987654321.
   (vi) If $V$ is a complex vector space and $X : V \to V$ is a linear map then there is a vector $v \in \mathbb{C}$ with $v \neq 0$ and $Xv = \lambda v$ for some $\lambda \in \mathbb{C}$.
   (vii) If $I_1$ and $I_2$ are ideals in a Lie algebra $L$ then $[I_1, I_2] \subseteq I_1 \cap I_2$.
   (viii) Any two representations of $sl(2,\mathbb{C})$ are isomorphic.
   (ix) Root spaces of a complex, semisimple Lie algebra are always one-dimensional.
   (x) There is a Lie algebra $L$ with Killing form $\kappa(\ ,\ )$ and base of simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ such that $(\alpha_1, \alpha_3) = 0$.

2. (a) (i) Define what it means for a vector space $L$ to be a Lie algebra.
   (ii) Define what it means for a subset $I$ of a Lie algebra $L$ to be an ideal.
   (iii) Define the derived algebra $L' = [L, L]$ of $L$ and prove that it is an ideal.
   (b) Determine the 3-dimensional complex Lie algebras $L$ for which $L' = [L, L] = \text{span}\{x\}$ is one-dimensional and $\text{span}\{x\} \subseteq Z(L)$.
   (c) For the Lie algebra
   $$L = \left\{ \begin{bmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathbb{C} \right\}$$
   of $3 \times 3$ matrices under commutation $[\ ,\ ]$, find $[L, L]$ and $Z(L)$ and hence determine if $L$ is
   (i) soluble
   (ii) nilpotent.
3. (a) (i) Define what it means for a Lie algebra to be nilpotent.

(ii) Using the definition show that if $L$ is nilpotent then $Z(L) \neq 0$.

(iii) Show that if $L/Z(L)$ is nilpotent then $L$ is nilpotent.

(iv) Does the result in (iii) result hold in general? That is, if $I$ is a nilpotent ideal and $L/I$ is also nilpotent is $L$ necessarily nilpotent? Give a reason for your answer.

(b) Let $L$ be a Lie algebra, $\text{Der}(L)$ the Lie algebra of derivations of $L$ and $\text{IDer}(L)$ the inner derivations of $L$.

Prove that:

(i) $\text{IDer}(L)$ is an ideal of $\text{Der}(L)$;

(ii) $L$ is nilpotent if and only if $\text{IDer}(L)$ is nilpotent;

(iii) if $D \in \text{Der}(L)$ and $[D, x] = 0$ for all $x \in L$ then $D(L) \subseteq Z(L)$.

4. (a) Let $L$ be a Lie algebra.

(i) Define what a representation of $L$ is.

(ii) Prove that every Lie algebra has a representation.

(iii) Define what it means for a bilinear form on $L$ to be invariant?

(iv) If $f : L \to gl(V)$ is a representation show that $( , )_f$ defined by $(x, y)_f = \text{tr}(f(x)f(y))$ is invariant. Recall that $gl(V)$ denotes the Lie algebra of all linear transformations from $V$ to $V$.

(b) Let $L = \mathfrak{sl}_2(\mathbb{C})$ be the algebra of all 2 by 2 matrices with zero trace. Recall that $\mathfrak{sl}(2, \mathbb{C})$ has basis $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(i) By calculation find $\kappa(h, e), \kappa(h, f)$ and $\kappa(e, f)$ where $\kappa( , )$ is the Killing form.

(ii) Give a Cartan subalgebra of $\mathfrak{sl}_2(\mathbb{C})$ and derive the corresponding root space decomposition.
5. (a) Let $L$ be a semisimple Lie algebra.

(i) Define what is meant by $H \subset L$ being a Cartan subalgebra and what the corresponding root space decomposition of $L$ is.

(ii) If roots $\alpha$ and $\beta$ satisfy $\alpha + \beta \neq 0$ and $x \in L_\alpha$ and $y \in L_\beta$ show that $\kappa(x, y) = 0$ where the $L_\alpha$ and $L_\beta$ are root spaces and $\kappa(\ , \ )$ is the Killing form.

(b) Let $H$ be a Cartan subalgebra for $L$ and $\{\alpha_1, \ldots, \alpha_r\}$ a base of simple roots for $L$.

(i) If there are $p$ positive roots what is the dimension of $L$?

(ii) Define the Cartan matrix of $L$.

(iii) Define the Dynkin diagram of $L$.

(c) If $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a base of simple roots with Cartan matrix

\[
\begin{bmatrix}
2 & -1 & -1 & -1 \\
-1 & 2 & 0 & 0 \\
-1 & 0 & 2 & 0 \\
-1 & 0 & 0 & 2 \\
\end{bmatrix},
\]

draw the corresponding Dynkin diagram.

(d) Suppose the semi-simple Lie algebra $L$ has Dynkin diagram

\[
\begin{tikzpicture}
  \node[shape=circle,draw] (a_1) at (0,0) {$\alpha_1$};
  \node[shape=circle,draw] (a_2) at (1,1) {$\alpha_2$};
  \node[shape=circle,draw] (a_3) at (2,2) {$\alpha_3$};
  \node[shape=circle,draw] (a_4) at (1,0) {$\alpha_4$};
  \draw (a_1) -- (a_2);
  \draw (a_2) -- (a_3);
  \draw (a_2) -- (a_4);
\end{tikzpicture}
\]

where $B = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a base of simple roots.

Find the corresponding Cartan matrix.