THE UNIVERSITY OF ADELAIDE AUSTRALIA

# Examination in School of Mathematical Sciences

## Semester 1, 2008

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| **016676** | **Honours Pure Mathematics — Lie Algebras IV** |
| | **PURE MTH 4005A** |

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| Official Reading Time: | 10 mins |
| Writing Time: | 180 mins |
| Total Duration: | 190 mins |

**NUMBER OF QUESTIONS: 5 TOTAL MARKS: 100**

**Instructions**

- Attempt all questions.
- Begin each answer on a new page.
- Examination materials must not be removed from the examination room.

**Materials**

- 1 Blue book is provided.
- Calculators are not permitted.

**DO NOT COMMENCE WRITING UNTIL INSTRUCTED TO DO SO.**

Each question is worth approximately the same number of marks.
In this exam all Lie algebras and vector spaces will be finite dimensional over the complex numbers $\mathbb{C}$.

1. Answer each of the following questions true or false. Give a short justification in each case. This might just be 'result proved in lectures' or the name of a theorem.
   (i) There exists a one-dimensional non-abelian Lie algebra.
   (ii) The derived algebra $L' = [L, L]$ of a Lie algebra $L$ is an ideal in $L$.
   (iii) The Killing form of a semisimple Lie algebra is non-degenerate.
   (iv) The kernel of a homomorphism of Lie algebras is not always an ideal.
   (v) There is no irreducible representation of $sl(2, \mathbb{C})$ of dimension 42.
   (vi) If $V$ is a complex vector space and $X : V \to V$ is a linear map then there is a vector $v \in \mathbb{C}$ with $v \neq 0$ and $Xv = \lambda v$ for some $\lambda \in \mathbb{C}$.
   (vii) If $I_1$ and $I_2$ are ideals in a Lie algebra $L$ then $[I_1, I_2] \subset I_1 \cap I_2$.
   (viii) Any two abelian Lie algebras are isomorphic.
   (ix) Root spaces of complex, semisimple Lie algebras are always one-dimensional.
   (x) There is a complex, semisimple Lie algebra $L$ with Killing form $\kappa(\ ,\ )$ and base of simple roots $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ such that $\kappa(\alpha_1, \alpha_2) = 2$.

2. (a) (i) Define what it means for a vector space $L$ to be a Lie algebra.
   (ii) If $L$ is a Lie algebra define what it means for a linear map $D : L \to L$ to be a derivation.
   (iii) Show that the set of all derivations $\mathrm{Der}(L) \subset gl(L)$ is a Lie subalgebra of $gl(L)$ if we give $gl(L)$ the Lie bracket which is the commutator of operators, that is $[X, Y] = XY - YX$ if $X, Y : L \to L$.
   (iv) Let $\mathrm{ad} : L \to gl(L)$ be the adjoint map defined by $\mathrm{ad}(x)(y) = [x, y]$. If we denote the image of ad in $gl(L)$ by $IDer(L)$ show that $IDer(L)$ is an ideal in $\mathrm{Der}(L)$. Give a condition on $L$ that will guarantee that ad is surjective (onto) ?

   (b) (i) Define the centre $Z(L)$ of a Lie algebra $L$.
   (ii) Show that the centre $Z(L)$ is an ideal of $L$.
   (iii) Let $L$ be a Lie algebra of dimension 3 in which $[L, L]$ is one-dimensional and $[L, L] \not\subset Z(L)$. Show that $L = L_2 \oplus Z(L)$ were $L_2$ is the non-abelian Lie algebra of dimension 2.

3. Let $L$ be a Lie algebra.
   (a) (i) Define what it means for $L$ to be nilpotent.
   (ii) Define what it means for $z \in gl(V)$ to be nilpotent.
   (iii) Show that if $x \in L$ and $L$ is nilpotent then $\mathrm{ad}(x) \in gl(L)$ is nilpotent.
   (iv) State the first version of Engel's theorem proved in lectures.
   (v) Using Engel's theorem prove that if $L$ is nilpotent then $L$ has a one-dimensional ideal.

   (b) (i) Define what it means for $L$ to be solvable.
   (ii) Assume that $L$ has ideals $0 \subset I_2 \subset I_1 \subset L$ such that $L/I_1$, $I_1/I_2$ and $I_2$ are all abelian. Show directly (ie not by just quoting the theorem from lectures) that $L$ is solvable.
   (iii) Give an example (without proof) of a Lie algebra which is solvable but not nilpotent.

4. (a) Let $L$ be a Lie algebra.
   (i) Define what it means for a symmetric, bilinear form $( \ , \ )$ on $L$ to be invariant.
   (ii) Show that the Killing form is invariant.
   (iii) Let $L = sl(2, \mathbb{C})$ the Lie algebra of all $2 \times 2$ traceless matrices. By calculation show that $(x, y) = \mathrm{tr}(xy)$ is a constant multiple of the Killing form $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$ and find the constant.

   (b) Let $L$ be a Lie algebra.
   (i) Define the radical of $L$.
   (ii) Define what it means for $L$ to be semisimple.
   (iii) State Cartan's criterion for $L$ to be semisimple.
   (iv) Show that if $L$ is semisimple then $Z(L) = 0$.

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5. (a) Let $L$ be a semisimple Lie algebra.

   (i) If $x \in L$ define what is meant by the abstract Jordan decomposition of $x$ and what it means for $x$ to be a semisimple element.

   (ii) Define what it means for $H \subset L$ to be a Cartan subalgebra and what the root space decomposition of $L$ is.

   (iii) For the Lie algebra $sl(3, \mathbb{C})$ of all $3 \times 3$ complex matrices with zero trace find a Cartan subalgebra and the root space decomposition.

   (b) Let $L$ be a semisimple Lie algebra with Killing form $\kappa( \ , \ )$.

   (i) If $B = \{\alpha_1, \ldots, \alpha_r\}$ is a base for the roots define the Cartan matrix of $L$ and the Dynkin diagram of $L$.

   (ii) Find the Dynkin diagram corresponding to the Cartan matrix:

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

   (iii) Find the Cartan matrix for the Dynkin diagram: