Lie Algebras IV 2008
Assignment 3. Solutions.

1. Let $x, y \in L$, $\alpha, \beta \in \mathbb{F}$, $\xi, \eta \in V^*$ and $v \in V$. First consider $(\alpha x + \beta y)(\xi)(v) = -\xi((\alpha x + \beta y)v) = -\xi(\alpha x v + \beta y v) = -\alpha\xi(xv) - \beta\xi(yv) = \alpha(x\xi)(v) + \beta(y\xi)(v) = (\alpha(x\xi) + \beta(y\xi))(v)$. Hence $(\alpha x + \beta y)(\xi) = \alpha(x\xi) + \beta(y\xi)$. [2] Second consider $(x(\alpha\xi + \beta\eta))(v) = -(\alpha\xi + \beta\eta)(xv) = -\alpha\xi(xv) - \beta\eta(xv) = \alpha x(\xi)(v) + \beta x(\eta)(xv) = (\alpha x(\xi) + \beta x(\eta))(v)$. Hence $x(\alpha\xi + \beta\eta) = \alpha x(\xi) + \beta x(\eta)$. [2] Finally $(x(y(\xi)) - y(x(\xi)))(v) = x(y(\xi))(v) - y(x(\xi))(v) = -y(\xi)(xv) + x(\xi)(yv) = \xi(yx(v)) - \xi(x(y(v)) = \xi(yx(v) - xy(v)) = \xi([y,x](v)) = -[y,x](\xi)(v) = [x,y](\xi)(v)$. Thus we have that $x(y(\xi)) - y(x(\xi)) = [x,y](\xi)$ [2] and we concluded that V^* with this action of L is an L-module.

[Marks: 2 + 2 + 2 = 6]

2. (a) Let $x, y \in L$, $v, w \in V$ and $\alpha, \beta \in \mathbb{F}$. Then $(\alpha x + \beta y) \star v = g(\alpha x + \beta y)g^{-1}v = g(\alpha x g^{-1}v + \beta y g^{-1}v) = \alpha(gxg^{-1}v) + \beta(gyg^{-1}v) = \alpha x \star v + \beta y \star v$ as required. [1] Also we have $x \star (\alpha v + \beta w) = gxg^{-1}(\alpha v + \beta w) = gx(\alpha g^{-1}v + \beta g^{-1}w) = g(\alpha x g^{-1}v + \beta x g^{-1}w) = \alpha gxg^{-1} + \beta gxg^{-1}w = \alpha x \star v + \beta x \star w$ as required. [1] Finally we check the bracket. $x \star (y \star v) - y \star (x \star v) = x \star (gyg^{-1}v) - y \star (gxg^{-1}v) = gxg^{-1}(gyg^{-1}v) - gyg^{-1}(gxg^{-1}v) = gxyg^{-1}v - gyxg^{-1}v = g([x,y](g^{-1}v)) = g[x,y]g^{-1}v = [x,y] \star v$. [1] So this is an L-module.

To see the module V_g is isomorphic to V we define $\theta\colon V \to V_g$ by $\theta(v) = gv$. It is straight forward to check that θ is a linear isomorphism of vector spaces [2]. Moreover $\theta(xv) = gxv = gxg^{-1}gv = x \star \theta(v)$ so that θ is an L-module isomorphism. [2]

(b) It is clear this map is linear so it would suffice to prove it is a homomorphism which can be done by a calculation. However it is easier to use part (a) as follows. Notice that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$$

because $a + d = 0$. It follows from part (a) that this is a representation and moreover it is isomorphic to the standard representation on $\mathbb{C}^2$. [2]

Notice that the defining representation is the usual representation where each matrix in $sl(2,\mathbb{C})$ is represented by itself or equivalently it corresponds the inclusion map $sl(2,\mathbb{C}) \to gl(2,\mathbb{C})$.

[Marks: 3 + 2 + 2 + 2 = 7]

3. $\mathrm{tr}([X,Y]Z) = \mathrm{tr}(XYZ) - \mathrm{tr}(YXZ) = \mathrm{tr}(XYZ) - \mathrm{tr}(XZY) = \mathrm{tr}(X[Y,Z])$. Notice to show that $\mathrm{tr}(YXZ) = \mathrm{tr}(XZY)$ it is enough to know that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ so that $\mathrm{tr}(Y(XZ)) = \mathrm{tr}((XZ)Y$. In general $\mathrm{tr}(X_1X_2\ldots X_n)$ is invariant under any *cyclic* permutation of $X_1, \ldots, X_n$ and $YXZ \mapsto XZY$ is a cyclic permutation.

[Marks: 2]

4. Let $x, y \in I^\perp$, $\alpha, \beta \in \mathbb{F}$ and $z \in I$. Note that $0 \in I^\perp$ as $\kappa(x, 0) = 0$. Also $\kappa(\alpha x + \beta y, z) = \alpha\kappa(x,z) + \beta\kappa(y,z) = \alpha 0 + \beta 0 = 0$ so that $\alpha x + \beta y \in I^\perp$ and hence $I^\perp$ is a subspace of L. [2] Now let $y \in L$ and consider $[y,x]$. We have $\kappa([y,x],z) = -\kappa([x,y],z) = -\kappa(x,[y,z]) = 0$ as $[y,z] \in I$. Thus $I^\perp$ is an ideal in L. [2]

Notice that you don't need to go back to the definition of the Killing form to prove any of this.

[Marks: 2 + 2 = 4]

5. $L^{(2)} = \mathrm{span}\{y,z\}$ so that $L^{(3)} = 0$. Hence L is solvable so the maximal solvable ideal is L. Hence $\mathrm{rad}(L) = L$.[2] We have $[x,y] = z$, $[x,z] = y$ and $[y,z] = 0$ so the matrices for the adjoint action relative to the basis $\{x,y,z\}$ are

$$\mathrm{ad}(x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \mathrm{ad}(y) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \text{ and } \mathrm{ad}(z) = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$\kappa(x,x) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(x)) = 2, \kappa(x,y) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y)) = 0, \kappa(x,z) = \mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(y) = 0$$

and

$$\kappa(y,y) = \mathrm{tr}(\mathrm{ad}(y)\,\mathrm{ad}(y)) = 0, \kappa(z,z) = \mathrm{tr}(\mathrm{ad}(z)\,\mathrm{ad}(z)) = 0, \kappa(y,z) = \mathrm{tr}(\mathrm{ad}(y)\,\mathrm{ad}(z)) = 0.$$

It follows that $L^\perp = \mathrm{span}\{y,z\} \neq \mathrm{rad}(L) = L$. [3]

Notice that you can also use Cartan's First Criterion to show that $L' \subset L^\perp$ which simplifies things.

[Marks: 2 + 3 = 5]

6. We have from lectures that if D is a derivation then $\mathrm{ad}(D(x)) = [D, \mathrm{ad}(x)]$. Hence $\kappa(D(x), y) = \mathrm{tr}(\mathrm{ad}(D(x))\,\mathrm{ad}(y)) = \mathrm{tr}([D, \mathrm{ad}(x)]\,\mathrm{ad}(y)) = -\mathrm{tr}([\mathrm{ad}(x), D]\,\mathrm{ad}(y)) = -\mathrm{tr}(\mathrm{ad}(x)[D, \mathrm{ad}(y)] = -\mathrm{tr}(\mathrm{ad}(x)\,\mathrm{ad}(D(y)) = -\kappa(x, D(y))$

[Marks: 3]

7. From the handout we have $\kappa(h, h) = 8$, $\kappa(e, f) = 4$ and every other inner product amongst the basis $\{e, h, f\}$ vanishes. It is straight forward to check that $(h, h) = 2$, $(e, f) = 1$ and all other combinations vanish. So by bilinearity of both κ and $(\, , \,)$ we conclude that $(x, y) = (1/4)\kappa(x, y)$ for all $x, y \in sl(2, \mathbb{C})$.

[Marks: 3]

Total marks 6 + 7 + 2 + 4 + 5 + 3 + 3 = 32