Lie Algebras IV 2009

Assignment 3. Due Wednesday 20th May 2009

1. Let V be an L-module and V^* the dual space of V. If $x \in L$ and $\xi \in V^*$ define $x\xi$ by $(x\xi)(v) = -\xi(xv)$ for all $v \in V$. Show that this makes V^* into an L-module.

2. Let L be a Lie algebra.

(a) Let V be an L-module. If $g \in GL(V)$ show that defining $x \star v = gxg^{-1}v$ makes V into a (new) L-module. Denote this L module by V_g and show that it is isomorphic to V. Note that V_g is obviously the same space as V it is just the action of L which is different.

(b) Consider the map $\chi\colon sl(2,\mathbb{C}) \to gl(2,\mathbb{C})$ defined by

$$\chi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$$

Show that this defines a representation of $sl(2,\mathbb{C})$ which is isomorphic to the defining representation of $sl(2,\mathbb{C})$ on $\mathbb{C}^2$. This means show that the two L-module structures on $\mathbb{C}^2$ are isomorphic.

3. If V is a finite dimensional vector space and X, Y and Z are linear maps from V to V show that $\operatorname{tr}([X,Y]Z) = \operatorname{tr}(X[Y,Z])$.

4. Let L be a Lie algebra with Killing form $\kappa(\ ,\ )$. If I is an ideal show that

$$I^\perp = \{x \in L \mid \kappa(x,y) = 0\ \forall\, y \in I\}$$

is also an ideal. Don't forget to check that $I^\perp$ is a vector subspace.

5. Consider the three-dimensional Lie algebra L defined by $[x,y] = z$, $[x,z] = y$ and $[y,z] = 0$. You don't need to prove this is a Lie algebra. Calculate $\operatorname{rad}(L)$, the Killing form and $L^\perp$. Hence show that $\operatorname{rad}(L)$ may not equal $L^\perp$.

6. Let L be a Lie algebra and $D\colon L \to L$ be a derivation. Show that

$$\kappa(D(x),y) + \kappa(x,D(y)) = 0$$

for all $x,y \in L$ where $\kappa(\ ,\ )$ is the Killing form. You may need a formula from Lecture 3 relating $\operatorname{ad}(D(x))$, $\operatorname{ad}(x)$ and D.

7. Let $L = sl(2,\mathbb{C})$ the Lie algebra of all 2×2 traceless matrices. By calculation show that $(x,y) = \operatorname{tr}(xy)$ is a constant multiple of the Killing form $\kappa(x,y)$ and find the constant.