Lie Algebras IV 2009

Assignment 2. Solutions.

1. We use the result from Assignment 1 that $L' = L_1' \oplus L_2'$ and induction to show that $L^{(k)} = L_1^{(k)} \oplus L_2^{(k)}$. For $k = 1$ we are done. Assume then that the result is true for $k \le m$. Then $L^{(m+1)} = (L^{(k)})' = (L_1^{(k)} \oplus L_2^{(k)})' = (L_1^{(k)})' \oplus (L_2^{(k)})' = L_1^{(k+1)} \oplus L_2^{(k+1)}$ [2]. If L is solvable then there is a k with $L^{(k)} = 0$ so that $0 = L^{(k)} = L_1^{(k)} \oplus L_2^{(k)}$ and thus $L_1^{(k)} = 0 = L_2^{(k)}$ so that L_1 and L_2 are solvable [1]. If L_1 and L_2 are solvable there exist k_1 and k_2 such that $L_1^{(k_1)} = 0$ and $L_2^{(k_2)} = 0$. If we let k be the maximum of k_1 and k_2 then $L_1^{(k)} = 0 = L_2^{(k)}$ so that $L^{(k)} = 0 \oplus 0 = 0$. Hence L is solvable [2].

[Marks: 2 + 1 + 2 = 5]

2. If L has an non-zero abelian ideal I then $I' = 0$ so that I is solvable [1]. If L has a non-zero solvable ideal I then there is a k such that $I^{(k-1)} \neq 0$ and $I^{(k)} = 0$. Let $J = I^{(k-1)} \neq 0$ then $[J, J] = I^{(k)} = 0$ so that J is a non-zero abelian ideal [2].

[Marks: 1+2 = 3]

3. We cannot have $\dim(Z(L)) = \dim(L)$ as then $L = Z(L)$ and L is abelian which is a contradiction [1]. So assume $\dim(Z(L)) = \dim(L) - 1$. Choose $y \in L$ with $y \notin Z(L)$. Then $\dim(Z(L) \oplus \operatorname{span}\{y\}) = \dim(Z(L)) + 1 = \dim(L)$ so that $L = Z(L) \oplus \operatorname{span}\{y\}$. So if x, x' are in L we can write them as $x = z + \alpha y$ and $x' = z' + \alpha' y$ for $z, z' \in Z(L)$ and $\alpha, \alpha' \in \mathbb{F}$. Now compute $[x, x'] = [z, z'] + \alpha[y, z'] + \alpha'[z, y] + \alpha\alpha'[y, y] = 0$. But then L is abelian which is a contradiction. Hence $\dim(Z(L)) \le \dim(L) - 2$ [3].

[Marks: 1 + 3 = 4]

4. (a) If $h \in I$ then we must also have $2e = [h, e] \in I$ and $2f = -[h, f] \in I$ so that I contains h, e and f and thus it contains $sl(2, \mathbb{C}) = \operatorname{span}\{h, e, f\}$ so that $I = sl(2, \mathbb{C})$. [2]

(b) If I is non-zero we can find $x = \alpha h + \beta e + \gamma f \in I$ with $x \neq 0$. If we bracket this with e, f and h we find that I must contain $-2\alpha e + \gamma h$, $2\alpha f - \beta h$ and $2\beta e - 2\gamma f$. Bracketing the first of these with f again gives $2\beta h \in I$ and similarly bracketing the second with e gives $2\gamma h \in I$. So if either of β or γ are non-zero then $h \in I$ and $I = sl(2, \mathbb{C})$. But if $\beta = \gamma = 0$ then, as $x \neq 0$, we must have $x = \alpha h \in I$ with $\alpha \neq 0$ so that $h \in I$ and again $I = sl(2, \mathbb{C})$. [3]

(c) We know that $sl(2, \mathbb{C})$ is not solvable and that $sl(2, \mathbb{C})$ has no other ideals besides 0. So $sl(2, \mathbb{C})$ has no solvable ideas except for 0. Hence the radical of $sl(2, \mathbb{C})$ is 0 and thus $sl(2, C)$ is semisimple. [2]

[Marks: 2 + 3 + 2 = 7]

5. If Z is an $n \times n$ matrix call the entries of the form X_{ij} with $j = i + d$ the dth diagonal. So the 0-th diagonal is what we usually think of as the diagonal, the $-(n-1)$-th diagonal is the entry X_{1n} in the bottom left-hand corner etc. From the results in Assignment 1 we can show that if X, Y have the $-n+1, -n+2, \ldots, d$ diagonals all zero then $[X, Y]$ has the $-n+1, -n+2, \ldots, d+1$ diagonals all zero. So the elements of $b(n, \mathbb{C})^{(k)}$ will have $-n+1, -n+2, \ldots, k-1$ diagonals all zero. Hence $b(n, \mathbb{C})^{(n)} = 0$. So $b(n, \mathbb{C})$ is solvable. [2]

However it is not nilpotent. To see this let E_{ij} $j > i$ be a basis for $b(n, \mathbb{C})'$, ie the strictly upper triangular matrices. Then we know that $[E_{ii}, E_{ij}] = E_{ij}$ so that $[b(n, \mathbb{C}), b(n, \mathbb{C}))'] = b(n, \mathbb{C})'$. Hence $b(n, \mathbb{C})^k = b(n, \mathbb{C})^1$ for all k. [2]

[Marks: 2 + 2 = 4]

6. Consider $x + L^{k+1} \in L^k / L^{k+1}$ where $x \in L^k$ and $y + L^{k+1}$ where $y \in L$. Then $[x + L^{k+1}, y + L^{k+1}] = [x, y] + L^{k+1} = 0$ because $[x, y] \in [L^k, L] = L^{k+1}$.

[Marks: 2]

7. ($\Rightarrow$) First we show linear independence. Assume that $\sum_{i=1}^{q} a_i v^i + V = 0$ in V/W. This means that $\sum_{i=1}^{q} a_i v^i \in W$ and hence $\sum_{i=1}^{q} a_i v^i = \sum_{j=1}^{p} b_j w^j$ for some $b_1, \ldots, b_p$ so that $\sum_{i=1}^{q} a_i v^i + \sum_{j=1}^{p} (-b_j) w^j = 0$. As $\{w^1, \ldots, w^p, v^1, \ldots, v^q\}$ is a basis of V this implies that $a_1 = \cdots = a_q = 0$ so that $\{w^1, \ldots, w^p\}$ are linearly

independent. [1] Second we show they span. Let $v + W \in V/W$ then $v = \sum_{j=1}^{p} b_i w^i + \sum_{i=1}^{q} a_i v^i$ for some $a_1, \dots, a_q, b_1, \dots, b_p$. Hence $v + V = \sum_{j=1}^{p} b_i w^i + \sum_{i=1}^{q} a_i v^i + V = \sum_{i=1}^{q} a_i (v^i + V)$ as required. [1]

($\Longleftarrow$) First we show linear independence. Assume that $\sum_{i=1}^{q} a_i v^i + \sum_{j=1}^{p} (-b_j) w^j = 0$. Then $0 = \sum_{i=1}^{q} a_i v^i + \sum_{j=1}^{p} (-b_j) w^j + W = \sum_{i=1}^{q} a_i (v^i + W)$ so that $a_1 = \cdots = a_q$ because $\{v^1 + W, \dots, v^q + W\}$ is a basis of V/W. Hence $\sum_{j=1}^{p} (-b_j) w^j = 0$ so that $b_1 = \cdots = b_p = 0$ because $\{w^1, \dots, w^p\}$ is a basis of W. This gives the required result. [1] Now let $v \in V$. Because $\{v^1 + W, \dots, v^q + W\}$ is a basis of V/W we there exist $a_1, \dots, a_q$ such that $v + V = \sum_{i=1}^{q} a_i (v^i + W) = \sum_{i=1}^{q} a_i v^i + W$ and hence $v - \sum_{i=1}^{q} a_i v^i \in W$. But then $\{w^1, \dots, w^p\}$ is a basis of W so there exists $b_1, \dots, b_p$ such that $v - \sum_{i=1}^{q} a_i v^i = \sum_{j=1}^{p} b_j w_j$. Thus $v = \sum_{i=1}^{q} a_i v^i + \sum_{j=1}^{p} b_j w^j$ as required. [1]

[Marks: 1+1+1+1 = 4]

8. We know that there exists $v \in V$, $v \neq 0$ such that $Xv = \lambda v$ for some $\lambda \in \mathbb{C}$. We prove the result by induction. If $\dim(V) = 1$ we are done as a 1×1 matrix is upper-triangular [1]. Assume the result is true whenever the dimension is less than k and consider V of dimension k [1]. Choose $v \neq 0$ so that $Xv = \lambda v$ and let $\mathbb{C}v$ be the line spanned by v. Then $X(\mathbb{C}v) \subset (\mathbb{C}v)$ so that there is a well-defined linear map $\bar{X}\colon V/\mathbb{C}v \to V/\mathbb{C}v$ given by $\bar{X}(w + \mathbb{C}v) = X(w) + \mathbb{C}v$ [1]. As $\dim(V/\mathbb{C}v) = k - 1$ we can apply the induction and find a basis $v_1 + \mathbb{C}v, \dots v_{k-1} + \mathbb{C}v$ for which $\bar{X}$ is upper-triangular [1]. From the previous question we know that $v, v_1, \dots, v_{k-1}$ is a basis of V [1]. We have that $X(v) = \lambda v$ and that $\bar{X}(v_i + \mathbb{C}v) = X(v_i) + \mathbb{C}v$ is a linear combination of the vectors $v_1 + \mathbb{C}v, \dots, v_i + \mathbb{C}v$ [1]. It follows that $X(v_i)$ is a linear combination of the vectors $v, v_1, \dots, v_i$ and thus X is upper-triangular. [1]

[Marks: 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7]