Lie Algebras IV 2008

Assignment 2. Due Wednesday 29th.

Note: I will be away for the two weeks of the break. Email might work.

1. Let the Lie algebra $L = L_1 \oplus L_2$ be a direct sum of two arbitrary Lie algebras L_1 and L_2. (This is not necessarily the L_2 defined in class.) Prove that L is solvable if and only if L_1 and L_2 are solvable.

2. Let L be a Lie algebra. Show that L has a non-zero solvable ideal if and only if L has a non-zero abelian ideal.

3. Let L be a non-abelian Lie algebra. Show that $\dim(Z(L)) \leq \dim(L) - 2$.

4. Consider the Lie algebra $sl(2,\mathbb{C})$ with basis h, e and f satisfying $[h,e] = 2e$, $[h,f] = -2f$ and $[e,f] = h$.

   a) Show that if I is an ideal in $sl(2,\mathbb{C})$ and $h \in I$ then $I = sl(2,\mathbb{C})$.

   b) Show that $sl(2,\mathbb{C})$ has no non-trivial ideas. (Hint: Let $x = \alpha h + \beta e + \gamma f \in I$ and try to use the fact that $[e,x]$, $[f,x]$ and $[h,x]$ are in I to show that $h \in I$.)

   c) Deduce from (b) that $sl(2,\mathbb{C})$ is semisimple.

5. Show that $b(n,\mathbb{C})$ the Lie algebra of $n \times n$, complex, upper triangular matrices is solvable but not nilpotent.

6. Let L be a Lie algebra and denote by L^k the lower central series. Show that $L^k/L^{k+1} \subseteq Z(L/L^{k+1})$.

7. Let W be a subspace of a finite-dimensional vector space V. Let $\{w^1,\dots,w^p\}$ be a basis of W. If $v^1,\dots,v^q$ are vectors in V show that $\{w^1,\dots,w^p,v^1,\dots,v^q\}$ is a basis of V if and only if $\{v^1+W,\dots,v^q+W\}$ is a basis of V/W.

8. Let V be a finite dimensional, complex, vector space and $x \in gl(V)$. Assume that $0 \neq v \in V$ is in the kernel of x and let $\mathbb{C}v$ be the span of v in V. Show that $\bar{x}\colon V/\mathbb{C}v \to V/\mathbb{C}v$, defined by $\bar{x}(w + \mathbb{C}v) = x(w) + \mathbb{C}v$ is well-defined. If $\{v^1 + \mathbb{C}v,\dots,v^r + \mathbb{C}v\}$ is a basis of $V/\mathbb{C}v$ with respect to which $\bar{x}$ is strictly upper triangular show that $\{v, v^1,\dots,v^r\}$ is a basis of V with respect to which x is strictly upper triangular.