Lie Algebras IV 2009

Assignment 1. Solutions.

1. (a) The simplest way of doing this is probably to draw two general matrices E_{ij} and E_{kl} and argue what happens. But in LaTeX it is easier to just use symbols so note that if the (p,q)th entry of E_{ij} is denoted $(E_{ij})^{pq}$ then we have $(E_{ij})^{pq} = \delta_{ip}\delta_{jq}$. So we have

$$(E_{ij}E_{kl})^{pq} = \sum_{r=1}^{n} \delta_{ip}\delta_{jr}\delta_{kr}\delta_{lq} = \delta_{jk}\delta_{ip}\delta_{lq} = \delta_{jk}(E_{il})^{pq}.$$

The result for the bracket now follows easily. [2]

(b) First the derived algebra. We know that as the E_{ij} are a basis of $gl_n(\mathbb{C})$ the derived algebra is spanned by all the brackets $[E_{ij}, E_{kl}]$. Moreover the trace of any bracket is zero so the derived algebra is inside the subalgebra of traceless matrices, that is $sl_n(\mathbb{C})$. If $i \neq l$ then $[E_{i1}, E_{1l}] = E_{il}$ is in the derived algebra. This means that all the matrices with zeroes on the diagonal are in the derived algebra. Now consider a diagonal matrix X with trace zero. As $X_{11} + \dots X_{nn} = 0$ we have $-X_{nn} = X_{11} + \dots X_{n-1n-1}$ and thus

$$\begin{aligned} X &= X_{11}(E_{11} - E_{22}) + (X_{11} + X_{22})(E_{22} - E_{33}) + \cdots + (X_{11} + X_{22} + \cdots + X_{nn})(X_{n-1n-1} - X_{nn}) \\ &= X_{11}[E_{12}, E_{21}] + (X_{11} + X_{22})[E_{23}, E_{32}] + \cdots + (X_{11} + X_{22} + \cdots + X_{nn})[X_{n-1n}, X_{nn-1}] \end{aligned}$$

It follows that every traceless matrix is in the derived algebra so that the derived algebra is $sl_n(\mathbb{C})$ [3].

Second the centre. First note that if X is a multiple of the identity matrix then $[X, Y] = 0$ for all $Y \in gl_n(\mathbb{C})$. We want to show that the converse is also true. Let X be in the centre then $[X, E_{ij}] = 0$ for all i and j. So

$$\sum_{k,l=1}^{n} X_{kl}\delta_{jk}E_{il} - X_{kl}\delta_{li}E_{kj} = 0$$

and taking the (p,q)th element and applying the sums to the Kronecker deltas gives

$$X_{jq}\delta_{pi} - X_{pi}\delta_{jq} = 0$$

for all i, j, p, q. Letting $p = i = q$ gives

$$X_{ji} - X_{ii}\delta_{ji} = 0$$

so if $i \neq j$ then $X_{ji} = 0$ and if $i = j$ then $X_{ii} = X_{jj}$. Hence X is a scalar multiple of the identity. There are other ways to prove this in particular you have $XA = AX$ for all A. So if A is an elementary matrix this means any column operation applied to X has the same result as the corresponding row operation. This forces X to be diagonal and then making all diagonal entries constant follows as above [2].

(c) Let $b_n(\mathbb{C})$ be the subset of all upper triangular matrices in $gl_n(\mathbb{C})$. Clearly $b_n(\mathbb{C})$ is a vector subspace. Assume A and B are upper triangular. So if $i > j$ then $A_{ij} = B_{ij} = 0$. Consider $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$. This can only be non-zero if $i \leq k$ and $k \leq j$ which forces $i \leq j$. Hence AB is upper triangular and so also is BA thus $[A, B]$ is upper triangular. We have thus shown that $b_n(\mathbb{C})$ is a Lie subalgebra of $gl_n(\mathbb{C})$. This is not an ideal. The best way to show this is to give example that works for every n. One such is $[E_{n1}, E_{11}] = E_{n1}$ but E_{n1} is not upper triangular. [3]

2. It is enough to show that for all $x, y \in L$ we have $[x, y]$ in the span of the set

$$\{[v_i, v_j] \mid 1 \leq i < j \leq n\}.$$

Let $x = \sum_{i=1}^{r} x_i v_i$ and $y = \sum_{j=1}^{r} y_j v_j$. Then

$$\begin{aligned} [x, y] &= \sum_{i,j=1}^{n} x_i y_j [v_i, v_j] \\ &= \sum_{i<j=1}^{n} (x_i y_j - x_j y_i)[v_i, v_j] \end{aligned}$$

So $\{[v_i, v_j] \mid 1 \leq i < j \leq n\}$ is a spanning set for L'. [3]

3. We have $\phi\colon L \to J$ an isomorphism of Lie algebras. Because ϕ is a bijection we can define $\phi^{-1}\colon J \to L$ which is well known to be a vector space isomorphism. We check it is a Lie algebra isomorphism. We have $\phi([\phi^{-1}(x), \phi^{-1}(y)]) = [x, y]$ so that $[\phi^{-1}(x), \phi^{-1}(y)] = \phi^{-1}([x, y])$. [1]

First the centre. Let $x \in Z(L)$ and $y \in J$ then $[x, \phi^{-1}(y)] = 0$ and applying ϕ gives $[\phi(x), y] = 0$ but this is true for all y so that $\phi(x) \in Z(J)$ or $\phi(Z(L)) \subset Z(J)$. Now swap L and J so that $\phi^{-1}(Z(J)) \subset Z(L)$ or $Z(J) \subset \phi(Z(L))$. Hence $Z(J) = \phi(Z(L))$ [3].

Secondly the derived algebra.

$$\begin{aligned} \phi(L') &= \phi\left(\operatorname{span}\{[x, y] \mid x, y \in L\}\right) \\ &= \operatorname{span}\{[\phi(x), \phi(y)] \mid x, y \in L\} \\ &= \operatorname{span}\{[u, v] \mid u, v \in J\} \qquad \text{(as } \phi \text{ is onto)} \\ &= J' \end{aligned}$$

Notice that at the first step here we use the following fact: If $f\colon V \to W$ is a linear map between vector spaces and $X \subset V$ then $f(\operatorname{span}(X)) = \operatorname{span}(f(X))$. [3]

4. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be in $L = L_1 \oplus L_2$ then $[x, y] = ([x_1, x_2], [y_1, y_2]) \in L_1' \oplus L_2'$. As $L_1' \oplus L_2'$ is a subalgebra we must have the that the subalgebra generated by all such $[x, y]$ is inside $L_1' \oplus L_2'$. That is $L' \subset L_1' \oplus L_2'$. On the other hand let $x_1, y_1 \in L_1$. Then $([x_1, y_1], 0) = [(x_1, 0), (y_1, 0)] \in L'$. Again this means that the subalgebra generated by all such elements is in L'. But that means that $L_1' \oplus 0 \subset L'$. Similarly $0 \oplus L_2' \subset L'$. As L' is a vector subspace the direct sum of these two spaces must also be contained in L' so $L_1' \oplus L_2' \subset L'$ and thus $L_1' \oplus L_2' = L'$. [4]

Now let $(x_1, x_2) \in Z(L_1 \oplus L_2)$. This happens if and only if $0 = [(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$ for all $(y_1, y_2) \in L_1 \oplus L_2$ which is if and only if $[x_1, y_1] = 0$ for all $y_1 \in L_1$ and $[x_2, y_2] = 0$ for all $y_2 \in L_2$ which is is if and only if $x_1 \in Z(L_1)$ and $x_2 \in Z(L_2)$. Thus $Z(L_1 \oplus L_2) = Z(L_1) \oplus Z(L_2)$. [4]

5. Let $a = a_x x + a_y y$ and $b = b_x x + b_y y$. Then $[a, b] = [a_x x + a_y y, b_x x + b_y y] = (a_x b_y - a_y b_x)x$ and we see the bracket is uniquely defined and anti-symmetric [2]. We check it satisfies the Jacobi identity. Let $c = c_x x + c_y c$. We have

$$\begin{aligned} [a, [b, c]] + [b, [c, a]] + [c, [[a, b]] &= [a, (b_x c_y - b_y c_x)x + [b, (c_x a_y - c_y a_x)x] + [c, (a_x b_y - a_y b_x)x] \\ &= (a_y(b_x c_y - b_y c_x) + b_y(c_x a_y - c_y a_x) + c_y(a_x b_y - a_y b_x))x \\ &= (a_y b_x c_y - c_y a_y b_x) + (-a_y b_y c_x + b_y c_x a_y) + (-b_y c_y a_x + c_y a_x b_y) \\ &= 0. \qquad [3] \end{aligned}$$

Hence L is a Lie algebra.

6. $\mathbb{C}^3$ is a complex vector space and the cross-product is a bilinear map $\mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$. It is well known that $x \times x = 0$ for all $x \in \mathbb{C}^3$. From first year we have

$$\begin{aligned} x \times (y \times z) &= y(x \cdot z) - z(x \cdot y) \\ y \times (z \times x) &= z(y \cdot x) - x(y \cdot z) \\ z \times (x \times z) &= x(z \cdot y) - y(z \cdot x) \end{aligned}$$

and clearly adding up gives the Jacobi identity. I was happy to take the equivalent result for tensors: $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ or a calculation with determinants. [5]

If $x \in Z(\mathbb{C}^3)$ cross-product x with the three usual basis vectors, actually two will do, to show that the three components of x are zero. [2]

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$. Then $e_1 \times e_2 = e_3$, $e_2 \times e_3 = e_1$ and $e_3 \times e_1 = e_2$. Thus $(\mathbb{C}^3)'$ contains e_1, e_2 and e_3 which span $\mathbb{C}^3$ so $(\mathbb{C}^3)' = \mathbb{C}^3$. [2]

Recall that $sl(2, \mathbb{C})$ has a basis e, f, h satisfying: $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. Let $H = (0, 0, 2i)$, $E = (1, i, 0)$ and $F = (-1, i, 0)$. It can be checked that $H \times E = 2E$, $H \times F = -2F$ and $E \times F = H$. It follows that the map sending $aH + bE + cF$ to $ah + be + ch$ is an isomorphism. This is the map:

$$(z_1, z_2, z_3) \mapsto \begin{bmatrix} \frac{1}{2i} z_3 & \frac{1}{2}(z_1 - iz_2) \\ \frac{-1}{2}(z_1 + iz_2) & -\frac{1}{2i} z_3 \end{bmatrix} \qquad [3]$$