Lie Algebras IV 2009

Assignment 1. Due Friday 27th March 2009.

1. Consider the Lie algebra $gl_n(\mathbb{C})$.

   a) Let E_{ij} be the matrix with a one in the (i,j) position and zeros elsewhere. Show that

   $$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$$

   where δ_{ij} is the Kronecker delta symbol which is one if $i = j$ and zero otherwise.

   b) Using part a) or otherwise find the derived algebra and centre of $gl_n(\mathbb{C})$. You might find it useful to compute things like: $[X, E_{ij}]$, for a general matrix X and $[E_{kk}, E_{ij}]$ and $[E_{ij}, E_{ji}]$.

   c) Show that the upper triangular matrices in $gl_n(\mathbb{C})$ are a Lie subalgebra. Are they an ideal ?

2. If L is a finite-dimensional Lie algebra with basis $v_1, \ldots, v_r$ show that $\{[v_i, v_j] \mid 1 \leq i < j \leq n\}$ is a spanning set for L'.

3. Let $\phi \colon L \to J$ be an isomorphism of Lie algebras. Show that $Z(J) = \phi(Z(L))$ and $J' = \phi(L')$.

4. Let $L = L_1 \oplus L_2$ where L_1 and L_2 are Lie algebras. Show that $L' = L_1' \oplus L_2'$ and $Z(L) = Z(L_1) \oplus Z_(L_2)$.

5. Let L be a two-dimensional vector space with basis x and y. Show that requiring bilinearity and antisymmetry and defining the bracket of x and y by $[x, y] = x$ suffices to define a unique Lie algebra L.

6. Show that $\mathbb{C}^3$ with the cross (vector) product as bracket is a Lie algebra. Calculate its centre and derived algebra. Find an explicit Lie algebra isomorphism between $\mathbb{C}^3$ and $sl(2, \mathbb{C})$.